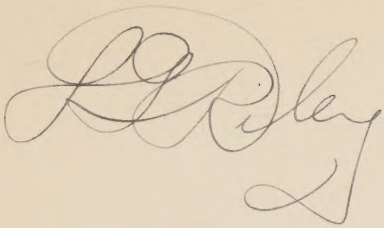
MANUAL AND KEY
TO PLANE GEOMETRY



MANUAL AND KEY
TO PLANE GEOMETRY

BY

WILLIAM W. STRADER

AND

LAWRENCE D. RHOADS

THE JOHN C. WINSTON COMPANY

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PREFACE

The authors hope that this Manual and Key may prove helpful to the busy or the inexperienced teacher. The **outlines of proof** will reduce the time required for the solution of all the written exercises. The **Suggestions** and **Notes** interspersed throughout not only give helpful advice for conducting a course of geometry, but also provide many directions which will aid the teacher in developing an individual teaching technique.

No effort has been made to present complete proofs except for some of the most important exercises. All that is attempted is to disclose the primary elements with which to build the proofs. These outlines are not to be regarded as models upon which to pattern proofs, nor are they necessarily the only or the shortest proofs possible. In some instances exercises, which have been previously proved, have been given as Facts to support steps. This has been done solely for the sake of brevity and teachers will have no difficulty in supplying the complete proofs. In class-work only those theorems which appear under the various section numbers of the text (the regular book theorems and corollaries) should be used as supporting Facts.

This Manual and Key is for teachers only. No teacher should permit a pupil to have access to it, nor should it be resold except to the publishers.

Criticisms and corrections will be thankfully received by the authors.

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MANUAL AND KEY TO PLANE GEOMETRY

THE PREFACE OF THE TEXT

Suggestion: Read in the Preface of the text, p. iii, paragraph 2; p. ix, first six lines; p. x, Miscellaneous Suggestions, paragraph 1.

THE INTRODUCTION

PAGE 3

Thought Exercises

Suggestion: These exercises were constructed to illustrate some of the processes necessary to the solution of problems of geometry. They should be used to provoke free discussion in class. Carelessness in grammatical construction, pronunciation, and thought should be checked at once by the teacher. Slow or listless pupils should be encouraged to enter into the discussions. Accept no part of an argument unless it can be supported by some common fact. Permit no diagrams, drawing, nor hand motions. Answers should be expressed in complete sentences. Pupils should answer in their own words and should not be required to give the answers exactly as listed here.

1. 345; 27,980; $(125)^4$. Numbers, whose right-hand figure is 0 or 5, are divisible by 5. Any product which has 5 as a factor is divisible by 5.

2. Each number of the second series (3 to 11) is two more than the corresponding numbers of the first series (1 to 9); the sum of any three numbers of the second series is six more than the sum of the corresponding numbers of the first series.

3. $\frac{18}{24} = \frac{90}{120}$; $\frac{12}{15} = \frac{96}{120}$ (reducing to equivalent fractions of the same denominator); $\frac{96}{120} - \frac{90}{120} = \frac{6}{120} = \frac{1}{20}$ (subtraction and reduction). $\frac{18}{24} = .75$; $\frac{12}{15} = .80$ (reducing to decimals); $.80 -$

.75 = .05 (subtraction). Each method shows the second fraction to be the greater.

4. 9,415 ÷ 7 = 1,345 (a product of two factors divided by one of them gives the other); 63,215 − 9,415 = 53,800 (a sum of two addends minus one gives the other); 53,800 ÷ 1,345 = 40 (first fact); 4 is the missing figure of the multiplier since the first place to the left of the unit's place is the ten's place.

5 and **6.** Numerous answers are possible. The answer for Ex. 5 will depend on use of the room, the placement of the windows, the intensity of the illumination, etc. The answer for Ex. 6 will depend upon the seating arrangement of the pupils, the order in which the papers are handled, etc. Accept any reasonable answer, if it can be defended and if it is properly and completely answered.

7. Be sure that the conditions of the problem are not violated. Permit no sketches and no hand motions. One answer might be: Thrust a loop of thin but strong cord through the neck and into the bottle. Roll the bottle about with the neck slightly tilted downward until the cork comes within the loop of the cord and the axis of the cork is approximately parallel with the axis of the bottle. In taking up any slack in the loop, the cord should be made to cross behind one circular end of the cork and remain in that position. A gentle but firm pull on the ends of the cord extracts the cork.

8. There are 100 days between July 4 and October 12. Since there are 7 days in one week, 100 days equals 14 weeks and 2 days. Two days after Monday is Wednesday. Pupils will readily see that this is a *proof* in that the result comes from facts already known.

9. 16 books are twice 8 books both in number and value. (Doubles of equals are equal.)

10. (*a*) The outside dimensions are $10'' \times 8''$ with an area of 80 sq. in. The inside dimensions are $6'' \times 8''$ with an area of 48 sq. in. (The area of a rectangle is the base times its altitude.) The area of the frame is the difference of these areas or 32 sq. in.

(*b*) The frame may be considered as made of two pieces each $1'' \times 10''$ and of two pieces each $1'' \times 6''$. $2 \times 1'' \times 10'' = 20$ sq. in. $2 \times 1'' \times 6'' = 12$ sq. in. (Area of a rectangle is the base times its

altitude.) Since the whole frame equals the sum of its parts, 20 sq. in. plus 12 sq. in. = 32 sq. in.

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Note on Definitions: There are only eleven vital concepts to be emphasized among the preliminary definitions given on pp. 5–20; they are found in §§ 20, 25, 26, 27, 28, 30, 33, 34, 35, 37, and 38. Pupils should know these definitions thoroughly; they should be able to sketch the required figures, if given their names; they should be able to name special figures and to state the relationships of parts which exist. **Teach pupils to state the relationship of parts whenever possible, by means of equations and not in words.**

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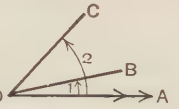
Exercises. Free-hand Drawing of Angles

Suggestion: Pupils should learn early to represent their diagrams or figures by a neat sketch or drawing. Actual constructions with the compasses and straightedge are unnecessary and their manipulation is often a hindrance rather than a help. The main purposes in such a group of exercises are to associate a figure with its name, thus aiding the pupil in visualization and to emphasize the important elements of the definitions of the various terms used. Care should be taken that the most general figure possible is drawn. Do not accept special cases. For example: in Ex. 2, do not permit *equal* adjacent angles; or in Ex. 3, two *right* angles. If the teacher desires, the pupil may write under each sketch the appropriate name or title. For example:

Ex. 7. $\measuredangle 1$ and 2 are supplementary adjacent angles.

Sometimes pupils are confused with the word *adjacent* when applied to angles. Show them by means of sketches the difference in meaning between *adjacent angles* and *angles adjacent to a side.* Later when Proposition 2 is discussed, state the theorem as given in the text; do not say, "a side and two adjacent angles." Thus,

you will obviate one of the difficulties which beginners often have.



The significance of *each* word in a definition may be impressed by showing that $\angle 1$ and $\angle 2$ are not adjacent angles, because, although they have the common vertex O and the common side OA, the common side is not *between* them.

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Exercises on Angles

1. 60°; 45°; 70° 45′; 64° 40′.

2. 90°; 135°; 149° 30′; 79° 10′.

3. $3x = 90°$; $x = 30°$; $2x = 60°$.

4. $x + 4x = 180°$; $x = 36°$; $4x = 144°$.

5. 30°; 60°; 90°.

6. $x = 10°$; $x + 70° = 80°$.

7. $x + (x - 120°) = 180°$; $x = 150°$; $x - 120° = 30°$.

8. 120° or 240°; $2\ (30°) - \frac{3}{4}\ (30°) = \frac{5}{4}\ (30°) = 37° 30'$. The reflex angle is 142° 30′.

9. 150°; 30°; 150°.

10. 72°.

11. 54°; 36°.

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Review Exercises

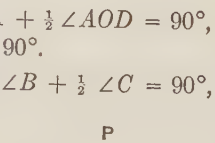
6. Any line through its mid-point is the bisector of a line.

7. An angle has one bisector.

8. 37° 30′.

9. $\angle BOA + \angle AOD = 180°$, $\frac{1}{2}\angle BOA + \frac{1}{2}\angle AOD = 90°$, $\angle 1 = \frac{1}{2}\angle BOA$, $\angle 2 = \frac{1}{2}\angle AOD$, $\angle 1 + \angle 2 = 90°$.

10. $\angle A + \angle B + \angle C = 180°$, $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90°$, $\frac{1}{2}\angle A = \angle 1$, $\frac{1}{2}\angle B = \angle 2$, $\frac{1}{2}\angle C = \angle 3$, $\angle 1 + \angle 2 + \angle 3 = 90°$.



11. $\angle 1 = \angle 2$; § 37.

12. $XS = YS$; § 37.

13. $OX = OY$ and $\angle 1 = \angle 2$; § 38.

PAGES 22–25

Note: §§ 40, 41, and their accompanying exercises may be omitted. They are not included in the Minimum Course suggested on pp. vii–viii of the text.

PAGE 26

Exercises. Principles of Geometry

Suggestion: The exercises of pp. 26, 28, and 29 may be used—

1. to familiarize the pupil with finding relationships of parts of figures;

2. to acquaint the pupil with the meaning and use of the Principles; and

3. to introduce the meaning of a proof.

Care taken here in being sure that the solutions are clear to each pupil and that the answers are complete and correct will pay dividends in future work. These exercises afford the teacher an opportunity o help the pupil think and talk geometry. Do not permit a pupil to say, for example: "Principle 1." He might say, "Any magnitude equals itself," or, "Identity," or, "Identical." Should the pupil express himself in "his own language," be sure the statement is simple, complete, and rigorous.

2. Statement A.

3. Statement A.

7. *Prin.* 1 (Identity).

9. *Prin.* 3 (Substitution).

10. *Prin.* 3.

14. $CD = BE$; *Prin.* 6 (Axiom of Subtraction).

20. *Prin.* 3.

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Exercises. Principles of Geometry

3. In aiming the gun, the two sights are "lined up" with a point on the target (*Prin.* 13).

6. DC is a straight line (Given). $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$ (*Prin.* 29); $\angle 1 =$ rt. $\angle$ (Given); $\angle 2 =$ rt. $\angle$ (*Prin.* 6).

8. *Prin.* 28.

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Exercises. Informal Proofs

Suggestion: Besides the three uses outlined for the exercises of pp. 26–29, these exercises offer an additional and an advanced step towards the Formal Proofs which are to come later. No attempt should be made to follow the parts and their order of a written demonstration as given in § 46. The words *Given* and *To prove* may be substituted for *Hypothesis* and *Conclusion* respectively, as is done elsewhere in the text. The latter words have been retained here to acquaint the pupils with their meanings. The suggestions previously given for the exercises on page 26 are pertinent to this group also. Call attention to the fact that the hypothesis always gives one or more relationships which are needed in the proof. **Have the pupils note also that the conclusion is the result of associating given, known, and derived parts.**

8. The *hypothesis* is that $\angle 1 + \angle x = 180°$, $\angle 2 + \angle y = 180°$, and $\angle 1 = \angle 2$. The *conclusion* is that $\angle x = \angle y$. The *proof* is that—

1. $\angle 1 + \angle x = 180°$.	1. Given.
2. $\angle 2 + \angle y = 180°$.	2. Given.
3. $\angle 1 + \angle x = 180° = \angle 2 + \angle y$.	3. Substitution.
4. $\angle 1 = \angle 2$.	4. Given.
5. $\angle x = \angle y$.	5. Axiom of Subtraction.

9. The *hypothesis* is that $\angle 1 = \angle 2$ and $\angle 1 + \angle 3 = 180°$. The *conclusion* is that $\angle 3 + \angle 2 = 180°$. The *proof* is that—

1. $\angle 1 + \angle 3 = 180°$.	1. Given.
2. $\angle 1 = \angle 2$.	2. Given.
3. $\angle 2 + \angle 3 = 180°$.	3. Substitution.

11. The pupil should now be able to appreciate the formal arrangement of a written demonstration. Call his attention to the manner in which the bisection of a line is indicated. As: $AD = DC = \frac{1}{2} AC$. Point out to him that although certain relationships are stated in the Given, they must be restated, as they are found necessary in the Proof. Emphasize that, when the relationship $AD = AE$ has been evolved, the Proof is finished.

Proof:

1. $AC = AB$.	1. Given.
2. $\frac{1}{2}\ AC = \frac{1}{2}\ AB$.	2. *Prin.* 8.
3. $\frac{1}{2}\ AC = AD$.	3. Given.
4. $\frac{1}{2}\ AB = AE$.	4. Given.
5. $AD = AE$.	5. Substitution.

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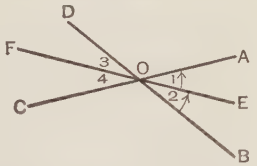
Practice Proofs

1. Theorem. The bisector of one of two vertical angles bisects the other.

Given: $\angle BOA$ and $\angle DOC$ are vertical angles and $\angle 1 = \angle 2 = \frac{1}{2}\angle BOA$.

To prove: $\angle 3 = \frac{1}{2}\angle 4$.

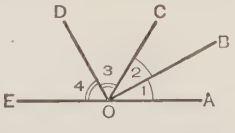


Proof: STATEMENTS	FACTS
1. $\angle BOA$ and $\angle DOC$ are vert. $\measuredangle$; EOF is a st. line.	1. Given.
2. BOD and AOC are st. lines.	2. § 35.
3. $\angle 1$ and $\angle 4$ and also $\angle 2$ and $\angle 3$ are vert. $\measuredangle$.	3. § 35.
4. $\angle 1 = \angle 2$.	4. Given.
5. $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$.	5. § 56.
6. $\angle 4 = \angle 3$.	6. Sub.

2. Theorem. The angle formed by the bisectors of a pair of supplementary adjacent angles is a right angle.

Given: $\angle AOC + \angle COE = 2$ rt. $\measuredangle$, $\angle 1 = \angle 2 = \frac{1}{2}\ \angle AOC$, and $\angle 3 = \angle 4 = \frac{1}{2}\ \angle COE$.

To prove: $\angle BOD =$ rt. $\angle$.



Proof: STATEMENTS	FACTS
1. $\angle AOC + \angle COE = 2$ rt. $\measuredangle$.	1. Given.
2. $\frac{1}{2}\angle AOC + \frac{1}{2}\angle COE =$ rt. $\angle$.	2. *Prin.* 8.
3. $\frac{1}{2}\angle AOC = \angle 2$, $\frac{1}{2}\angle COE = \angle 3$.	3. Given.
4. $\angle 2 + \angle 3 =$ rt. $\angle$.	4. Sub.
5. $\angle BOD = \angle 2 + \angle 3$.	5. *Prin.* 4.
6. $\angle BOD =$ rt. $\angle$.	6. Sub.

3. Theorem. (Omitted.)

Given: $CO \perp OA$, $DO \perp OB$, and $\angle AOR = \angle ROB = \frac{1}{2} \angle AOB$.

To prove: $\angle COR = \angle ROD$.

Proof:	STATEMENTS	FACTS
1.	$CO \perp OA$, $DO \perp OB$.	1. Given.
2.	$\angle COA$ and $\angle BOD$ are rt. $\measuredangle$.	2. § 27.
3.	$\angle COA = \angle BOD$.	3. *Prin.* 26.
4.	$\angle AOR = \angle ROB$.	4. Given.
5.	$\angle COA + \angle AOR = \angle BOD + \angle ROB$.	5. *Prin.* 5.
6.	$\angle COA + \angle AOR = \angle COR$ and $\angle BOD + \angle ROB = \angle ROD$.	6. *Prin.* 4.
7.	$\angle COR = \angle ROD$.	7. Sub.

4. Theorem. (Omitted.)

Given: $CO \perp OA$, $DO \perp OB$, and ROS bisects $\angle LAOB$, making $\angle AOR = \angle ROB = \frac{1}{2} \angle AOB$.

To prove: $\angle ROB + \angle SOC =$ rt. $\angle$.

Proof:	STATEMENTS	FACTS
1.	ROS is a st. line.	1. Given.
2.	$\angle AOR + \angle COA + \angle SOC = 180°$.	2. § 54.
3.	$CO \perp OA$.	3. Given.
4.	$\angle COA = 90°$.	4. §§ 27, 36.
5.	$\angle AOR + \angle SOC = 90°$.	5. *Prin.* 6.
6.	$\angle AOR = \angle ROB$.	6. Given.
7.	$\angle ROB + \angle SOC = 90°$.	7. Sub.

5. Theorem. The exterior sides of supplementary adjacent angles form a straight line.

Given: $\angle 1 + \angle 2 = 180°$.

To prove: AOC is a st. line.

Proof:	STATEMENTS	FACTS
1.	$\angle 1 + \angle 2 = 180°$.	1. Given.
2.	$\angle 1 + \angle 2 = \angle AOC$.	2. *Prin.* 4.
3.	$\angle AOC = 180°$.	3. Sub.
4.	Line AOC is a st. line.	4. § 28.

CHAPTER ONE

Note 1: Teachers should stress particularly those propositions and corollaries which are marked **C** and **N.** Such theorems and problems are considered of prime importance by the most eminent mathematical authorities. They constitute the foundations upon which various courses in geometry may be built. To them may be added the "**c** and **n** theorems" and the exercises which are best fitted to meet the given conditions under which each teacher is working. It is absolutely imperative, therefore, that **no teacher should simply teach the text page after page.** Selection of the proper material must be made and this may vary with the nature of the course, the time allotted to the subject, and the ability of the pupils in the class. The National Committee has this to say: "The committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry." For this reason, the text offers an unexcelled variety of material for selection. On this point read also, in the Preface of the text, the last half of p. vi and the first half of p. vii.

Note 2: If one were given two triangular pieces of cardboard and wished to find if they were of the same size and shape, it would be perfectly natural to place one upon the other to see if they would match. The word *congruent* is derived from the Latin word, *congruo,* which means "to agree in every particular." Thus, the fundamental character of superposition precludes the discarding of the proofs as given in the text for Propositions 1 and 2. As long as the College Entrance Examination Board and such a committee as the National Committee require these proofs, the teacher should not ignore them. There is a distinct advantage, however, in omitting full and rigorous proofs of these propositions *at the*

beginning of a course and in postulating the truths of these theorems, because of the infrequency of proofs by superposition and because of their unsatisfying nature to beginners. Pupils so frequently begin their formal work in geometry by committing to memory the proofs of these first two propositions, thus making an unprofitable and an improper start. Teachers of experience find that where the method of proof by superposition is used in the early lessons pupils try to prove everthing in sight by this method. It may seem better to have pupils commit the *statements of the theorems* to memory and to emphasize their importance as tools by solving the simple exercises given in the text, especially those on pp. 44 and 48.

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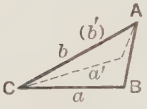
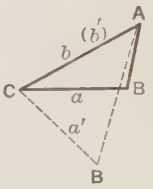
Exercises

1.





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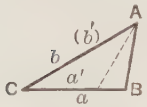

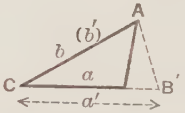

3. The triangles are congruent by § 65, (*s.a.s.* = *s.a.s.*).

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Sight Exercises. Corollary Method

1. In $\triangle AOD$ and AOB,

1. $AD = AB$.	1. Given.
2. $\angle 3 = \angle 4$.	2. Given.
3. $DO = OB$.	3. Given.
4. $\triangle AOD \cong \triangle AOB$.	4. *s.a.s.* = *s.a.s.*

2. In $\triangle DOC$ and COB,

1. $DO = OB$.	1. Given.
2. $\angle 5 = \angle 6$.	2. Given.
3. $CO = CO$.	3. Iden.
4. $\triangle DOC \cong \triangle COB$.	4. *s.a.s.* = *s.a.s.*

3. In $\triangle AOD$ and AOB,

1. $AD = AB$.	1. Given.
2. $\angle 1 = \angle 2$.	2. Given.
3. $AO = AO$.	3. Iden.
4. $\triangle AOD \cong \triangle AOB$.	4. *s.a.s.* = *s.a.s.*

4. In $\triangle AOD$ and AOB,

1. $DO = OB$.	1. Given.
2. $AO \perp DB$.	2. Given.
3. $\angle 7$ and $\angle BOA$ are rt. $\angle$s.	3. § 27.
4. $\angle 7 = \angle BOA$.	4. *Prin.* 26.
5. $AO = AO$.	5. Iden.
6. $\triangle AOD \cong \triangle AOB$.	6. *s.a.s.* = *s.a.s.*

5. In $\triangle AOD$ and COB,

1. $OA = OC$.	1. Given.
2. $OD = OB$.	2. Given.
3. $\angle 7 = \angle 6$.	3. *Prin.* 27.
4. $\triangle AOD \cong \triangle COB$.	4. *s.a.s.* = *s.a.s.*

6. Same proof as Ex. 5.

7.

1. $\triangle ADB$ is isos.	1. Given.
2. $AD = AB$.	2. § 61.
3. $\angle 1 = \angle 2$.	3. Given.
4. $AO = AO$.	4. Iden.
5. $\triangle AOD \cong \triangle AOB$.	5. *s.a.s.* = *s.a.s.*

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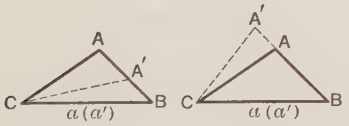
Exercises

1.



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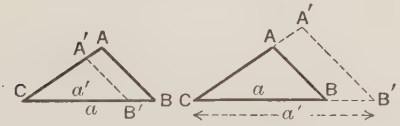

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Sight Exercises. Congruent Triangles

1. $AD = CD$ (Given); $\angle ADB = \angle BDC$ (Given); $BD = BD$ (Iden.); $\triangle ADB \cong \triangle CDB$ (*s.a.s.* = *s.a.s.*).

2. $\angle ABD = \angle DBC$ (Given); $\angle BDA = \angle CDB$ (Given); $DB = DB$ (Iden.); $\triangle ABD \cong \triangle CBD$ (*a.s.a.* = *a.s.a.*).

3. Call O the point of bisection; $AO = OC$ (Given); $BO = OD$ (Given); $\angle BOA = \angle DOC$ (*Prin.* 27); $\triangle BOA \cong \triangle DOC$ (*s.a.s.* = *s.a.s.*).

4. Call O the mid-point of AB; $AO = OB$ (Given); $\angle A = \angle B$ (Given); $\angle DOA = \angle COB$ (*Prin.* 27); $\triangle DOA \cong \triangle COB$ (*a.s.a.* = *a.s.a.*).

5. $BC = DA$ (Given); $\angle B = \angle D$ (Given); $\angle C = \angle A$ (Given); $\triangle BOC \cong \triangle DOA$ (*a.s.a.* = *a.s.a.*).

6. $\triangle ABC$ is isos. (Given); $AB = AC$ (§ 61); $\angle DAB = \angle CAD$ (Given); $AD = AD$ (Iden.); $\triangle DAB \cong \triangle CAD$ (*s.a.s.* = *s.a.s.*).

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Suggestions for a Recitation:

1. Have a pupil read the last four lines of p. 49; then have these lines read in unison by the class; follow this by asking different pupils to state in their own words the four steps in proving two lines (or two angles) equal.

2. Have Ex. 1, p. 50, read silently by the class.

3. Develop the proof in some such manner as indicated by the following questions: What is to be proved? In which triangles are these lines parts? How many pairs of equal parts must be known to prove the congruence of the triangles? Name one pair of equal parts; why are they equal? Name a second pair; why are they equal? Name a third pair; why are they equal? Why are the triangles congruent? Why are AB and CB corresponding sides? Why are they equal? Who can name all the facts we have used to build our proof? What other pupil will name them? Could Proposition 2 (*a.s.a.* = *a.s.a.*) be used to prove $\triangle ABD$ congruent to $\triangle CBD$? Why?

PAGE 50

Exercises. Corresponding Parts

1. $AD = CD$ (Given); $\angle BDA = \angle CDB$ (Given); $BD = BD$ (Iden.); $\triangle ADB \cong \triangle CDB$ (*s.a.s.* = *s.a.s.*); $AB = BC$ (*Prin.* 38).

2. $\angle BDA = \angle CDB$ (Given); $\angle ABD = \angle DBC$ (Given); $BD = BD$ (Iden.); $\triangle ABD \cong \triangle CBD$ (*a.s.a.* = *a.s.a.*); $AB = BC$ and $AD = DC$ (*Prin.* 38).

3. Let O be the point of intersection of AC and DB: $AO = OC$ and $DO = OB$ (Given); $\angle AOD = \angle COB$ and $\angle BOA = \angle DOC$ (*Prin.* 27); $\triangle AOD \cong \triangle COB$ and $\triangle BOA \cong \triangle DOC$ (*s.a.s.* = *s.a.s.*); $AD = BC$ and $AB = DC$ (*Prin.* 38).

4. $AO = OB$ (Given); $\angle A = \angle B$ (Given); $\angle AOD = \angle BOC$ (*Prin.* 27); $\triangle AOD \cong \triangle BOC$ (*a.s.a.* = *a.s.a.*); $AD = CB$, $\angle D = \angle C$, and $CO = OD$ (*Prin.* 38).

PAGE 53

Exercises. Equal Angles

1. $AB = AD$ (Given); $CB = CD$ (Given); $\angle ABD = \angle BDA$ (§ 80); $\angle DBC = \angle CDB$ (§ 80); $\angle ABD + \angle DBC = \angle BDA + \angle CDB$ (*Prin.* 5); $\angle B = \angle ABD + \angle DBC$, $\angle D = \angle BDA + \angle CDB$ (*Prin.* 4); $\angle B = \angle D$ (Sub.).

2. Draw DB and prove $\angle D = \angle B$ by the method of Ex. 1. Draw AC and prove $\angle A = \angle C$ by the same method.

3. $AB = AC$ (Given); $\frac{1}{2} AB = \frac{1}{2} AC$ (*Prin.* 8); $AD = \frac{1}{2} AB$ (Given); $AE = \frac{1}{2} AC$ (Given); $AD = AE$ (Sub.); $\angle 1 = \angle 2$ (§ 80).

PAGE 55

Exercises. Congruent Triangles

1. $\triangle abc$ and $a'b'c'$ are equilateral (Given); $a = b = c$, $a' = b' = c'$ (§ 61); $a = a'$ (Given); $b = b', c = c'$ (Sub.); $\triangle abc \cong \triangle a'b'c'$ (*s.s.s.* = *s.s.s.*).

2. The like marked lines indicate that $AB = BC = CD = AD$ (Given); $DB = DB$ (Iden.); $\triangle BAD \cong \triangle BCD$ (*s.s.s.* = *s.s.s.*).

3. $AC = BD$ (Given); $AD = BC$ (Given); $DC = DC$ (Iden.); $\triangle ADC \cong \triangle BCD$ (*s.s.s.* = *s.s.s.*).

4. $AB = CD$ (Given); $BO = OD$ (Given); $AO = OC$ (Given); $\triangle OBA \cong \triangle ODC$ (*s.s.s.* = *s.s.s.*); or, $OA = OC$ (Given); $BO = OD$ (Given); $\angle BOA = \angle DOC$ (*Prin.* 27); $\triangle OBA \cong \triangle ODC$ (*s.a.s.* = *s.a.s.*).

5. Have pupils give *full* statements of §§ 65, 71, and 84.

PAGE 55

Auxiliary Construction Lines

Suggestion: It may be advisable to call attention to the fundamental character of Principle 13, "Two points determine a line," since elementary constructions depend upon this principle alone. It may be of doubtful value to confuse pupils by telling them that, "A line may be drawn equal to a given line and bisecting a given angle," or, "a line may be drawn equal to a given line and be either perpendicular or parallel to a second given line." Should a pupil suggest either statement himself, it will not be difficult for the teacher to show him that two constructions are really necessary to determine such a line. Call the attention of pupils to such statements as, "Draw line AD" and explain that points A and D are thus named through which the line is drawn. Emphasize § 87; if pupils thoroughly understand its meaning, many troublesome difficulties will be avoided.

PAGE 56

Exercises. Conditions of Construction

1. AD passes through A and $AD \perp CD$. The construction is possible.

2. AD passes through A and through the mid-point of CB. Possible.

3. AD passes through A and makes $\angle CAD = \angle DAB$. Possible.

4. The line must pass through A, must pass through the mid-point of CB, and be perpendicular to CB. Impossible.

5. The line must pass through A, divide $\angle A$ into two equal parts, and be perpendicular to BC. Impossible.

6. The line must pass through the mid-point of YZ, through O, and through X. The construction is impossible, even though O

be the center of an equilateral triangle. One of the conditions which the line must satisfy would require proof and could not be assumed when the line is drawn.

PAGE 58

Suggestion: The group of exercises on pp. 58, 59 offers an excellent opportunity for drill on the three most important cases of congruence. Call the attention of pupils to the fact that three pairs of equalities are needed to prove congruence and that these are found in the statements of the Given and in special relationships of parts which exist in the figures. Several of the exercises offer drill in proving congruence through right triangles. Such proofs should be considered as secondary to the three cases of general congruence. Point out the significance of § 88, p. 57, and that such relationships as are expressed in (*a*) to (*f*) of that section are best stated as equations, since it is in the equational form that they are used in proofs. Emphasize that no matter which case of congruence is used, at least one pair of equalities must involve equal sides. At this point, it is valuable to ask pupils to compose original theorems of their own; the figures on pp. 58, 59 lend themselves for such purpose.

PAGE 58

Sight Exercises. Congruent Triangles

1. $AO = OC$ (Given); $AB \perp AD$; $CD \perp BC$ (Given); $\angle A = \angle C =$ rt. $\angle$ (§ 27, *Prin.* 26); $\angle AOB = \angle DOC$ (*Prin.* 27); $\triangle AOB \cong \triangle COD$ (*a.s.a.* = *a.s.a.*). Or use *l.a.* = *l.a.* (§ 72).

2. $AC = AB$ (Given); $\angle 1 = \angle 2$ (Given); $AD = AD$ (Iden.); $\triangle ACD \cong \triangle ABD$ (*s.a.s.* = *s.a.s.*). Or use *a.s.a.* = *a.s.a.*

3. $BD = BC$ (Given); $DE = EC$ (Given); $BE = BE$ (Iden.); $\triangle BDE \cong \triangle BCE$ (*s.s.s.* = *s.s.s.*). Or use *s.a.s.* = *s.a.s.*

4. $ST = TR$ (Given); $PT \perp RS$ (Given); $\angle RTP = \angle PTS =$ rt. $\angle$ (§ 27, *Prin.* 26); $PT = PT$ (Iden.); $\triangle PTS \cong \triangle PTR$ (*s.a.s.* = *s.a.s.*). Or use *l.l.* = *l.l.* (§ 69).

5. $CH = CH$ (Iden.); $CH \perp DE$ (Given); $\angle CHD = \angle CHE =$ rt. $\angle$ (§ 27, *Prin.* 26); $\angle 1 = \angle 2$ (Given); $\triangle CHE \cong \triangle CHD$ (*a.s.a.* = *a.s.a.*). Or use *l.a.* = *l.a.* (§ 72).

6. $AC = AC$ (Iden.); $\angle CAB = \angle DAC$, $\angle BCA = \angle ACD$ (Given); $\triangle ADC \cong \triangle ABC$ (*a.s.a. = a.s.a.*).

7. $BO = OC$ (Given); $AB \perp BC$, $CD \perp BC$ (Given); $\angle OBA = \angle OCD$ = rt. $\angle$ (§ 27, *Prin.* 26); $\angle AOB = \angle DOC$ (*Prin.* 27); $\triangle ABO \cong \triangle DCO$ (*a.s.a. = a.s.a.*). Or use *l.a. = l.a.* (§ 72).

8. $AO = OD$ (Given); $BO = OC$ (Given); $\angle BOA = \angle COD$ (*Prin.* 27); $\triangle ABO \cong \triangle CDO$ (*s.a.s. = s.a.s.*).

9. $AB = CD$ (Given); $\angle 1 = \angle 2$ (Given); $DB = DB$ (Iden.); $\triangle ABD \cong \triangle BCD$ (*s.a.s. = s.a.s.*).

10. $AB = AE$ (Given); $BC = ED$ (Given); $\angle B = \angle E$ (Given); $\triangle ABC \cong \triangle ADE$ (*s.a.s. = s.a.s.*).

11. $AO = CO$ (Given); $BO = DO$ (Given); $\angle AOB = \angle DOC$ (*Prin.* 27); $\triangle ABO \cong \triangle CDO$ (*s.a.s. = s.a.s.*).

12. $PS = RT$ (Given); $\angle 2 = \angle 1$ (Given; $TS = TS$ (Iden.); $\triangle PST \cong \triangle RTS$ (*s.a.s. = s.a.s.*).

13. $AC = AB$ (Given); $\frac{1}{2}\ AC = \frac{1}{2}\ AB$ (*Prin.* 8); $\frac{1}{2}\ AC = AD = DC$ (Given); $\frac{1}{2}\ AB = AE = EB$ (Given); $AD = AE$ (Sub.); $\angle A = \angle A$ (Iden.); $\triangle ABD \cong \triangle ACE$ (*s.a.s. = s.a.s.*); $EB = DC$ (Sub.); $\angle B = \angle C$ (§ 80); $CB = CB$ (Iden.); $\triangle BCE \cong BCD$ (*s.a.s. = s.a.s.*).

14. $BD = BC$ (Given); $\angle C = \angle D$ (§ 80); $\frac{1}{2}\angle C = \frac{1}{2}\angle D$ (*Prin.* 8); $\frac{1}{2}\angle C = \angle BCG = \angle GCD$ (Given); $\frac{1}{2}\angle D = \angle HDB = \angle CDH$ (Given); $\angle BCG = \angle HDB$ (Sub.); $\angle B = \angle B$ (Iden.); $\triangle BCG \cong \triangle BDH$ (*a.s.a. = a.s.a.*); $\angle GCD = \angle CDH$ (Sub.); $DC = DC$ (Iden.); $\triangle CDG \cong \triangle DCH$ (*a.s.a. = a.s.a.*).

15. This exercise has the same solution as No. 3.

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Exercises. Equal Lines and Equal Angles

1. Use a proof similar to that of Ex. 3, p. 58, and $\angle 1 = \angle 2$ (*Prin.* 38).

2. Use the proof of Ex. 2 of the preceding group and $CD = DB$ (*Prin.* 38).

3. Use either proof of Ex. 14 of the preceding group and $DH = CG$ (*Prin.* 38).

4. The proof corresponds to that of Ex. 13 of the preceding group. Take $ZS = YR$ (*Prin.* 38).

5. $\triangle ABC \cong \triangle EFG$ (Given); $AB = EF$ (*Prin.* 38); $\frac{1}{2}\,AB = \frac{1}{2}\,EF$ (*Prin.* 8); $\frac{1}{2}\,AB = AD$ (Given); $\frac{1}{2}\,EF = EH$ (Given); $AD = EH$ (Sub.); $\angle A = \angle E$ (*Prin.* 38); $AC = EG$ (*Prin.* 38); $\triangle ADC \cong \triangle EHG$ (*s.a.s. = s.a.s.*); $CD = GH$ (*Prin.* 38).

6. Use the figure of Ex. 5; $\angle C = \angle G$ (*Prin.* 38); $\frac{1}{2}\,\angle C = \frac{1}{2}\,\angle G$ (*Prin.* 8); $\frac{1}{2}\,\angle C = \angle DCA$ (Given); $\frac{1}{2}\,\angle G = \angle HGE$ (Given); $\angle DCA = \angle HGE$ (Sub.); $AC = GE$ (*Prin.* 38); $\angle A = \angle E$ (*Prin.* 38); $\triangle ADC \cong \triangle EHG$ (*a.s.a. = a.s.a.*); $CD = GH$ (*Prin.* 38).

7. Locate a point O within sight of M and N. Measure MO and NO and extend each through O, making $OR = OM$ and $OP = ON$. $\angle POR = \angle NOM$ (*Prin.* 27); $\triangle POR \cong \triangle NOM$ (*s.a.s. = s.a.s.*); $PR = MN$ (*Prin.* 38).

8. $AB = AD$ (Given); $CB = CD$ (Given); $AC = AC$ (Iden.); $\triangle ABC \cong \triangle ADC$ (*s.s.s. = s.s.s.*); $\angle BCA = \angle ACD$ (*Prin.* 38); $\angle CAB = \angle DAC$ (*Prin.* 38).

PAGE 60

Two-step Proofs

Suggestion: The group of exercises which follows may be reserved for review or they may be assigned as "extra credit exercises." Before assigning them, call attention to § 89 and explain its meaning. It will be found that Exs. 5 and 6 of the preceding group offer preliminary practice in using corresponding parts of congruent triangles to build proofs of the congruence of other triangles.

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Exercises. Two-step Proofs

1. $AB = AD$ (Given); $BC = CD$ (Given); $AC = AC$ (Iden.); $\triangle ABC \cong \triangle ADC$ (*s.s.s. = s.s.s.*); $\angle OAB = \angle DAO$ (*Prin.* 38); $AO = AO$ (Iden.); $\triangle ABO \cong ADO$ (*s.a.s. = s.a.s.*). § 83 may also be used.

2. $AC = DB$, $AB = DC$ (Given); $CB = CB$ (Iden.); $\triangle ABC \cong \triangle DCB$ (*s.s.s. = s.s.s.*); $\angle ABC = \angle BCD$, $\angle BCA = \angle DBC$

(*Prin.* 38); $\angle 2 = \angle 1$ (*Prin.* 6); $\angle A = \angle D$ (*Prin.* 38); $\triangle AOB \cong \triangle DOC$ (*a.s.a.* = *a.s.a.*).

3. $AB = AD$, $BO = OD$ (Given); $AO = AO$ (Iden.); $\triangle AOB \cong \triangle AOD$ (*s.s.s.* = *s.s.s.*); $\angle BOA = \angle AOD$ (*Prin.* 38); $\angle BOA = \angle DOC$ (*Prin.* 27); $\angle AOD = \angle COB$ (*Prin.* 27); $\angle DOC = \angle COB$ (Sub.); $CO = CO$ (Iden.); $\triangle BOC \cong \triangle DOC$ (*s.a.s.* = *s.a.s.*).

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Proposition 6

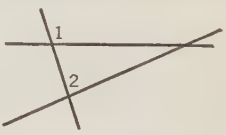
Suggestion: Teachers are cautioned not to put too much emphasis upon the proofs of Propositions 6 and 7 and all others which are marked with **c** and **n.** For further explanation, read the report of the National Committee and the syllabus of the College Entrance Examination Board.

Do not proceed so slowly that your work will be rushed at the end of the term. Allow plenty of time for review. **Remember that not all of the exercises are to be taken.** Select certain exercises according to the course, the class, and the individuals. Rapid-fire oral recitations can be made to cover a great many geometric facts in a short space of time; they also stimulate the slower thinkers in a class.

PAGE 62

Angles Formed by a Transversal

Note: Attention is called to the definitions of the angles formed by a transversal to two other lines not necessarily parallel lines. Although these definitions are simply worded and are rigorous, they should not be overemphasized by the teacher. Since some syllabi still use "corresponding angles" in place of *exterior-interior angles*, pupils should be familiar with both names. However, the term "corresponding" is neither appropriate, suggestive, nor descriptive, as may be seen in the annexed figure, in which $\angle 1$ and $\angle 2$ are *exterior-interior angles.* It is only when a and b are parallel that the term "corresponding" has any significance. Even then there are reasonable objections to its use. **Pupils should realize that all alternate**



interior angles (and all exterior-interior angles) are not equal. When the theorems of §§ 101, 102, and 104 are quoted, be sure that the words "of parallel lines" are included.

In order that pupils may get plenty of practice in recognizing the angles defined in §§ 92–96 in connection with parallel lines, figures similar to the following may be used. Pupils should be asked to name the various pairs of angles.

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Sight Exercises. Substitution Method

Suggestion: Before assigning these exercises, show pupils how they may "get started" by following (1) of § 103; teach them that a proof consists in bringing about the statement of the conclusion.

1. $\angle 2 = \angle 3$ (*Prin.* 27); $a \parallel b$ (Given); $\angle 3 = \angle 7$ (§ 102); $\angle 2 = \angle 7$ (Sub.).

2. $\angle 1 + \angle 3 = 180°$ (*Prin.* 29); $a \parallel b$ (Given); $\angle 3 = \angle 7$ (§ 102); $\angle 1 + \angle 7 = 180°$ (Sub.).

3. $a \parallel b$ (Given); $\angle 2 = \angle 6$, $\angle 5 = \angle 1$ (§ 102); $\angle 2 = 2\ \angle 5$ (Given); $\angle 6 = 2\ \angle 1$ (Sub.).

4. $c \parallel d$ (Given); $\angle 2 = \angle 3$ (§ 102); $a \parallel b$ (Given); $\angle 3 = \angle 7$ (§ 102); $\angle 2 = \angle 3 = \angle 7$ (Sub.).

5. $a \parallel b$ (Given); $\angle 3 = \angle 7$ (§ 102); $c \parallel d$ (Given); $\angle 7 = \angle 5$ (§ 102); $\angle 3 = \angle 7 = \angle 5$ (Sub.).

6. $a \parallel b$ (Given); $\angle 1 = \angle 4$ (§ 102); $c \parallel d$ (Given); $\angle 4 = \angle 6$ (§ 102); $\angle 6 = \angle 8$ (*Prin.* 27); $\angle 1 = \angle 4 = \angle 6 = \angle 8$ (Sub.).

7. $c \parallel d$ (Given); $\angle 5 + \angle 6 = 180°$ (§ 104); $a \parallel b$ (Given); $\angle 2 = \angle 5$ (§ 102); $\angle 2 + \angle 6 = 180°$ (Sub.).

8. $PS = PR$ (Given); $\angle R = \angle S$ (§ 80); $GH \parallel PS$ (Given); $\angle S = \angle 1$ (§ 102); $\angle R = \angle S = \angle 1$ (Sub.).

9. $PS = PR = RS$ (Given); $\angle P = \angle R = \angle S$ (§ 81); $GH \parallel PS$ (Given); $\angle S = \angle 1$ (§ 102); $\angle P = \angle S = \angle 1$ (Sub.).

10. $GH \parallel PS$ (Given); $\angle P + \angle PGH = 180°$ (§ 104); use the proof of Ex. 9 to show $\angle P = \angle 1$; $\angle 1 + \angle PGH = 180°$ (Sub.).

11. $BA = BO$ (Given); $\angle A = \angle 1$ (§ 80); $\angle 1 = \angle 2$ (*Prin.* 27); $CO = CD$ (Given); $\angle 2 = \angle D$ (§ 80); $\angle A = \angle 1 = \angle 2 = \angle D$ (Sub.).

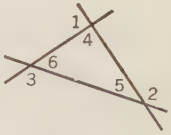
12. $CD = OC$ (Given); $2\,OC = BO$ (Given); $OC = \frac{1}{2}\,BO$ (*Prin.* 8); $BO = AB$ (Given); $CD = \frac{1}{2}\,AB$ (Sub.).

PAGE 71

Sight Exercises

1. $d \parallel b$ (Cons.); $\angle 4 + \angle 3 = \angle 1$ (§ 101); $\angle 4 = \angle 2$ (§ 101); $\angle 2 + \angle 3 = \angle 1$ (Sub.).

2. $\angle 1 = \angle 6 + \angle 5$ (§ 110); $\angle 2 = \angle 4 + \angle 6$ (§ 110); $\angle 3 = \angle 5 + \angle 4$ (§ 110); $\angle 1 + \angle 2 + \angle 3 = 2\,\angle 6 + 2\,\angle 5 + 2\,\angle 4$ (*Prin.* 5); $\angle 6 + \angle 5 + \angle 4 = 180°$ (§ 106); $2\,\angle 6 + 2\,\angle 5 + 2\,\angle 4 = 360°$ (*Prin.* 7); $\angle 1 + \angle 2 + \angle 3 = 360°$ (Sub.).



3. $\angle 3 + \angle 4 = 180°$ (*Prin.* 29); $\angle 4 = 140°$ (Given); $\angle 3 = 40°$ (*Prin.* 6); $\angle 1 + \angle 2 = \angle 4$ (§ 110); $\angle 1 + \angle 2 = 140°$ (Sub.); $\angle 2 = 100°$ (Given); $\angle 1 = 40°$ (*Prin.* 6); $\angle 3 = 40° = \angle 1$ (Sub.).

4. Let the three angles be represented by $2\,x$, $2\,x$, and $5\,x$; $2\,x + 2\,x + 5\,x = 180°$ (§ 106); $9\,x = 180°$ or $x = 20°$; $2\,x = 40°$; $5\,x = 100°$. The angles are 40°, 40°, and 100°.

5. Have pupils give *complete sentences* of Facts 1, 2, 4, 5, 6, 7, 8, 9, and 10 found listed in the Summary of Chapter One, p. 116.

General Suggestion

As in Ex. 5, it will be found excellent review, from time to time, to have pupils write out the Facts associated with the various kinds or classes of theorems. The Summaries at the close of the chapters will be found helpful for such work.

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Numerical Calculations

Note: The exercises on p. 73 and others of like nature should be given careful attention because of the arithmetic and algebraic

character of their solutions. Various kinds of numerical exercises are appearing more and more in recent examinations because they test pupils both in geometric facts and in the applications of algebraic technique.

If an exercise is divided into a number of parts, such as Exs. 2, 3, 4, and 5, do not assign all the parts of an exercise in one lesson. Plan your assignments so that the duller pupils may learn by progressive daily repetitions.

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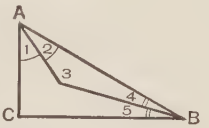
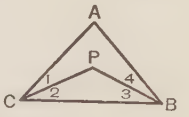
Exercises. Angles of Triangles

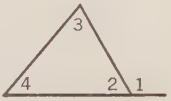


1. $\angle C$ = rt. $\angle$ (Given); $\angle A + \angle B = 90°$ (§ 108); $\frac{1}{2}\angle A + \frac{1}{2}\angle B = 45°$ (*Prin.* 8); $\frac{1}{2}\angle A = \angle 2$, $\frac{1}{2}\angle B = \angle 4$ (Given); $\angle 2 + \angle 4 = 45°$ (Sub.); $\angle 2 + \angle 3 + \angle 4 = 180°$ (§ 106); $\angle 3 = 135°$ (*Prin.* 6).



2. $180° - \angle P = \angle 2 + \angle 3$ (§ 107); $\angle P = 180° - (\angle 2 + \angle 3)$ (*Prin.* 6); $\angle 2 = \frac{1}{2}\angle C$, $\angle 3 = \frac{1}{2}\angle B$ (Given); $\angle 2 + \angle 3 = \frac{1}{2}\angle B + \frac{1}{2}\angle C$ (*Prin.* 5); $\angle B + \angle C = 180° - \angle A$ (§ 107); $\frac{1}{2}\angle B + \frac{1}{2}\angle C = 90° - \frac{1}{2}\angle A$ (*Prin.* 8); $\angle P = 180° - (90° - \frac{1}{2}\angle A)$ (Sub.) or $\angle P = 90° + \frac{1}{2}\angle A$; $\angle A = 40°$ (Given); $\angle P = 90° + 20° = 110°$ (Sub.); in like manner, the other values of $\angle P$ are 120°, 130°, $112\frac{1}{2}°$, $\frac{1}{2}(180° + x°)$.

3. (*a*) $x + 2x + 3x = 180°$ (§ 106); $6x = 180°$; $x = 30° = \angle A$; $2x = 60° = \angle B$; $3x = 90° = \angle C$. (*b*) $x + 3x + 5x = 180$; 20°; 60°; 100°. (*c*) $x + \frac{1}{2}x + \frac{1}{4}x = 180$; $102\frac{6}{7}°$; $51\frac{3}{7}°$; $25\frac{5}{7}°$. (*d*) $x + 2x + \frac{1}{2}x = 180$; $51\frac{3}{7}°$; $102\frac{6}{7}°$; $25\frac{5}{7}°$. (*e*) $x + (x + 20) + (x + 40)° = 180°$; 40°; 60°; 80°. (*f*) $x + (x - 30) + (x + 60) = 180°$; 50°; 20°; 110°. (*g*) $x + (2x - 5) + (3x + 35) = 180°$; 25°; 45°; 110°.



4. $\angle 3 + \angle 4 = \angle 1$ (§ 110); $\angle 1 = 120°$, $\angle 4 = 50°$ (Given); $\angle 3 + 50° = 120°$ (Sub.) or $\angle 3 = 70°$; $\angle 1 + \angle 2 = 180°$ (*Prin.* 29); $120° + \angle 2 = 180°$ (Sub.) or $\angle 2 = 60°$; in like manner, the other values are $\angle 3 = 50°$, $\angle 2 = 50°$; $\angle 3 = 70°$, $\angle 2 = 57\frac{1}{2}°$.

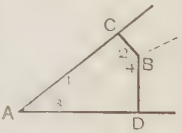
5. $x + y = 100°$; $x - y = 30°$; $2x = 130°$; $x = 65°$, $y = 35°$, $z = 80°$; in like manner the other values are $x = 70°$, $y = 10°$, $z = 100°$; $x = 40°$, $y = 2\frac{1}{2}°$, $z = 137\frac{1}{2}°$; $x = 39\frac{1}{2}°$, $y = 13\frac{1}{2}°$, $z = 127°$.

6. $AB = AC$ (Given); $\angle B = \angle C$ (§ 80); $\angle B = \angle C = \frac{5}{9}$ rt. $\angle = 50°$ (Given); $A = 180° - (\angle B + \angle C)$ (§ 107); $A = 180° - (100°) = 80°$ (Sub.).

7. $AB = AC$ (Given); $\angle 4 = \angle 5$ (§ 80); $\angle 1 = \angle 2 = \frac{1}{2}\ \angle BAD$ (Given); $\angle 1 + \angle 2 = \angle 4 + \angle 5$ (§ 110); $\angle 1 + \angle 1 = \angle 5 + \angle 5$ (Sub.); $2\ \angle 1 = 2\ \angle 5$ or $\angle 1 = \angle 5$ (*Prin.* 8); $\angle 4 = \angle 5 = \angle 1 = \frac{1}{2}\ \angle BAD$ (Sub.).

8. $\angle 1 = \angle 7 + \angle 8$ (§ 110); $\angle 7 + \angle 3 + \angle 4 = 180°$ and $\angle 8 + \angle 5 + \angle 6 = 180°$ (§ 54); $\angle 3 = \angle 4$, $\angle 5 = \angle 6$ (Given); $\angle 7 + 2\ \angle 3 = 180°$ and $\angle 8 + 2\ \angle 5 = 180°$ (Sub.); $\angle 7 + \angle 8 + 2\ \angle 3 + 2\ \angle 5 = 360°$ (*Prin.* 5); $\angle 1 + 2\ \angle 3 + 2\ \angle 5 = 360°$ (Sub.); $\angle 3 + \angle 5 = 180° - \angle 2$ (§ 107); $2\ \angle 3 + 2\ \angle 5 = 360° - 2\ \angle 2$ (*Prin.* 7); $\angle 1 = 2\ \angle 2$ (*Prin.* 6); $\angle 2 = \frac{1}{2}\ \angle 1$ (*Prin.* 8).

9. $BC \perp AC$, $BD \perp AD$ (Given); draw AB (*Prin.* 13); $\angle 1 + \angle 2 = 90°$, $\angle 3 + \angle 4 = 90°$ (§ 108); $\angle 1 + \angle 3 + \angle 2 + \angle 4 = 180°$ (*Prin.* 5); or $\angle A + \angle B = 180°$.



10. The sum of all the angles about A, B, and $C = 3 \times 360° = 1080°$ (§ 36, *Prin.* 7); the sum of the interior angles A, B, and $C = 180°$ (§ 106); the sum of all the angles without the triangle $= 900° = 5 \times 180°$ (*Prin.* 6).

11. Suggest to pupils that one method of composing the statement of an original exercise is (1) to draw a figure, assuming for it some special relationships of parts; (2) to find what special relationship results; (3) to combine the descriptions of (1) and (2) in a sentence. Guard against loose or careless composition.

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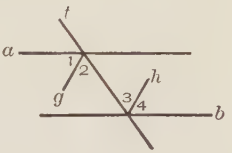
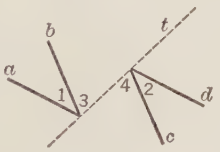
Exercises. Parallel Lines

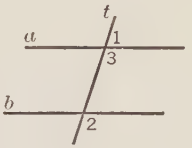
1. $a \parallel b$ (Given); $\angle 1 + \angle 2 = \angle 3 + \angle 4$ (§ 101); $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (Given); $2\ \angle 2 = 2\ \angle 3$ (Sub.); $\angle 2 = \angle 3$ (*Prin.* 8); $g \parallel h$ (§ 114).



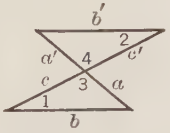
2. Draw t (*Prin.* 13); $b \parallel c$ (Given); $\angle 3 = \angle 4$ (§ 101); $\angle 1 = \angle 2$ (Given); $\angle 3 + \angle 1 = \angle 4 + \angle 2$ (*Prin.* 5); $a \parallel d$ (§ 114).



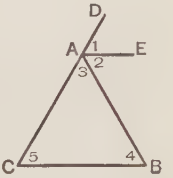
3. $\angle 1 + \angle 2 = 180°$ (Given); $\angle 1 + \angle 3 = 180°$ (*Prin.* 29); $\angle 3 = \angle 2$ (*Prin.* 28); $a \parallel b$ (§ 116).



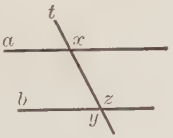
4. $a = a'$, $c = c'$ (Given); $\angle 3 = \angle 4$ (*Prin.* 27); $\triangle abc \cong \angle a'b'c'$ (*s.a.s.* = *s.a.s.*); $\angle 1 = \angle 2$ (*Prin.* 38); $b \parallel b'$ (§ 114).



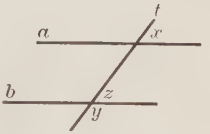
5. $\angle 1 + \angle 2 = \angle 4 + \angle 5$ (§ 110); $\angle 1 = \angle 2$, $\angle 4 = \angle 5$ (Given); $2\ \angle 2 = 2\ \angle 4$ (Sub.); $\angle 2 = \angle 4$ (*Prin.* 8); $AE \parallel CB$ (§ 114).



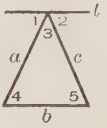
6. $a \parallel b$ (Given); $\angle x = \angle z$ (§ 102); $\angle z = \angle y$ (*Prin.* 27); $\angle x = \angle z = \angle y$ (Sub.).



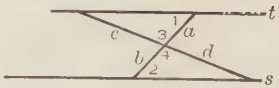
7. $a \parallel b$ (Given); $\angle x = \angle z$ (§ 102); $\angle z + \angle y = 180°$ (*Prin.* 29); $\angle x + \angle y = 180°$ (Sub.).



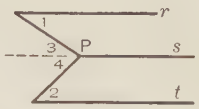
8. $a = c$ (Given); $\angle 4 = \angle 5$ (§ 80); $\angle 1 = \angle 2$ (Given); $\angle 4 + \angle 5 = 180° - \angle 3$ (§ 107); $\angle 1 + \angle 2 + \angle 3 = 180°$ (§ 54); $\angle 1 + \angle 2 = 180° - \angle 3$ (*Prin.* 6); $\angle 4 + \angle 5 = \angle 1 + \angle 2$ (Sub.); $2\ \angle 5 = 2\ \angle 2$ (Sub.); $\angle 5 = \angle 2$ (*Prin.* 8); $t \parallel b$ (§ 114).



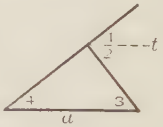
9. $a = b$ (Given); $\angle 3 = \angle 4$ (*Prin.* 27); $t \parallel s$ (Given); $\angle 1 = \angle 2$ (§ 101); $\triangle act \cong \triangle bds$ (*a.s.a.* = *a.s.a.*); $c = d$ (*Prin.* 38).



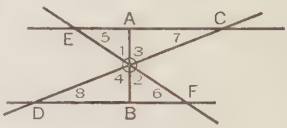
10. Produce s through P (*Prin.* 11); $r \parallel s$ (Given); $\angle 1 = \angle 3$ (§ 101); $s \parallel t$ (Given); $\angle 2 = \angle 4$ (§ 101); $\angle 1 + \angle 2 = \angle 3 + \angle 4$ (*Prin.* 5); $\angle P = \angle 3 + \angle 4$ (*Prin.* 4); $\angle 1 + \angle 2 = \angle P$ (Sub.).



11. The figure suggests that t is drawn parallel to a (*Prin.* 14); $\angle 1 = \angle 4$ (§ 102); $\angle 2 = \angle 3$ (§ 101); $\angle 1 + \angle 2 = \angle 3 + \angle 4$ (*Prin.* 5).



12. $EC \parallel DF$ (Given); $\angle 5 = \angle 6$ (§ 101); $OA = OB$ (Given); $\angle 1 = \angle 2$ (*Prin.* 27); $\triangle OAE \cong \triangle OBF$ (*s.a.a.* = *s.a.a.*); $AE = BF$ (*Prin.* 38); similarly $AC = BD$; $AE + AC = BF + BD$ (*Prin.* 5).



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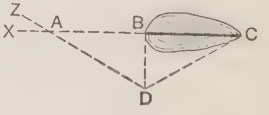
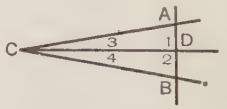
Exercises. Isosceles Triangles

1. $AC = AB$ (Given); $AD = AE$ (Given); $\angle DAC = \angle BAE$ (*Prin.* 27); $\triangle DAC \cong \triangle EAB$ (*s.a.s.* = *s.a.s.*); $DC = EB$ (*Prin.* 38).

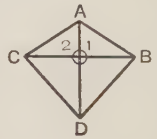
2. Sight a line through C and B such as CBX. At B turn a right angle and on BD take D within sight of C. From BD at D turn $\angle BDZ = \angle CDB$; $\triangle DBC \cong \triangle DBA$ (*a.l.* = *a.l.*); $BC = AB$ (*Prin.* 38).



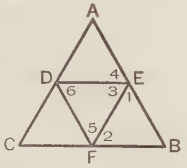
3. $\angle 1 = \angle 2$ (§ 27, *Prin.* 26); $\angle 3 = \angle 4$ (Given); $CD = CD$ (Iden.); $\triangle CDA \cong \triangle CDB$ (*a.s.a.* = *a.s.a.*); $CA = CB$ (*Prin.* 38); $\triangle CAB$ is isos. (§ 61).



4. $\triangle AOB \cong \triangle AOC$ by a two-step proof similar to that of Ex. 1, p. 60. $\angle 1 = \angle 2$, $CO = OB$ (*Prin.* 38); $AOD \perp COB$ (§ 26).



5. $AB = BC = AC$ (Given); $\angle A = \angle B = \angle C$ (§ 81); $\angle A + \angle B + \angle C = 180°$ (§ 106); $3\ \angle A = 180°$ (Sub.); $\angle A = \angle B = \angle C = 60°$ (*Prin.* 8 and Sub.). $\frac{1}{2}\ AB = \frac{1}{2}\ BC$ (*Prin.* 8); $\frac{1}{2}\ AB = BE$, $\frac{1}{2}\ BC = BF$ (Given); $BE = BF$ (Sub.); $\angle 1 = \angle 2$ (§ 80); $\angle 1 + \angle 2 + \angle B = 180°$ (§ 106); $2\ \angle 1 + 60° = 180°$ (Sub.); $\angle 1 = \angle 2 = 60° = \angle B$ (Sub.); $\triangle BEF$ is equilateral (§ 121). In like manner, $\triangle DCF$ and $\triangle ADE$ are equiangular and equilateral. $\angle 1 + \angle 3 + \angle 4 = 180°$ (§ 54); $60° + \angle 3 + 60° = 180°$ (Sub.); $\angle 3 = 60°$ (*Prin.* 6); and so also $\angle 5 = \angle 6 = 60°$; $\triangle DEF$ is equilateral (§ 121).



6. $AB = AC$ (Given); $\frac{1}{3}\ AB = \frac{1}{3}\ AC$ (*Prin.* 8); $\frac{1}{3}\ AB = AG = GH$, $\frac{1}{3}\ AC = AE = ED$ (Given); $AG = GH = AE = ED$ (Sub.); $AG + GH = AE + ED$ (*Prin.* 5); $\angle A = \angle A$ (Iden.); $\triangle AEH \cong \triangle AGD$ (*s.a.s.* = *s.a.s.*); $EH = DG$ (*Prin.* 38).

7. There are the following five cases. 1. base and base angle = base and base angle (*a.s.a.* = *a.s.a.*); 2. base and angle at vertex = base and angle at vertex (*s.a.a.* = *s.a.a.*); 3. leg and adjacent base angle = leg and adj. base angle (*s.a.a.* = *s.a.a.*); 4. leg and opposite base angle = leg and opp. base angle (*s.a.a.* = *s.a.a.*); 5. leg and angle at vertex = leg and angle at vertex (*s.a.s.* = *s.a.s.*).

Note: Any one proof should be accepted as a proper recitation from a pupil. As an "extra credit assignment," all proofs should be expected.

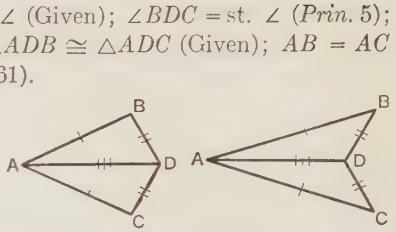
8. Take $AM = MB$ (*Prin.* 12); draw PM (*Prin.* 13); $PM = PM$ (Iden.); $PA = PB$ (Given); $\triangle PMA \cong \triangle PMB$ (*s.s.s.* = *s.s.s.*); $\angle 1 = \angle 2$ (*Prin.* 38); $PM \perp AB$ (§ 26); PM is the perpendicular bisector of AB (§ 38).

Note: Ex. 8 is given later in the text, p. 102, where it appears with a family of theorems. Pupils should understand that since P is in PM, P lies in the perpendicular bisector.

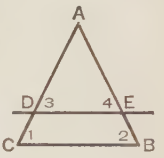
9. $AB = BC = AC$ (Given); $\angle A = \angle B = \angle C$ (§ 81); $AX = BY = CZ$ (Given); $XB = YC = ZA$ (*Prin.* 6); $\triangle AXZ \cong \triangle BYX \cong \triangle CZY$ (*s.a.s.* = *s.a.s.*); $XZ = YX = ZY$ (*Prin.* 38).

10. $\angle BDA = \angle ADC =$ rt. $\angle$ (Given); $\angle BDC =$ st. $\angle$ (*Prin.* 5); BDC is a st. line (*Prin.* 30); $\triangle ADB \cong \triangle ADC$ (Given); $AB = AC$ (*Prin.* 38); $\triangle ABC$ is isos. (§ 61).

Note: It is necessary to show BDC is a straight line in order to know that ABC is a triangle and not a quadrilateral such as



11. $DE \parallel CB$ (Given); $\angle 3 = \angle 1$, $\angle 4 = \angle 2$ (§ 102); $AB = AC$ (Given); $\angle 1 = \angle 2$ (§ 80); $\angle 3 = \angle 4$ (Sub.); $AD = AE$ (§ 120).



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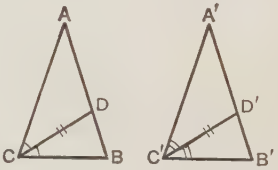
Review Exercises. Congruent Triangles

1. $\angle 1 + \angle CDA =$ st. $\angle$, $\angle 4 + \angle BCD =$ st. $\angle$ (*Prin.* 29); $\angle 1 = \angle 4$ (Given); $\angle CDA = \angle BCD$ (*Prin.* 28); $\angle 2 = \angle 3$ (Given); $\angle CDB = \angle ACD$ (*Prin.* 6); $DC = DC$ (Iden.); $\triangle ADC \cong \triangle BCD$ (*a.s.a. = a.s.a.*).

2. $\angle 1 + \angle DAO =$ st. $\angle = \angle 2 + \angle OBC$ (*Prin.* 29, Sub.); $\angle 1 = \angle 2$ (Given); $\angle DAO = \angle OBC$ (*Prin.* 28); $DO = OC$ (Given); $\angle AOD = \angle COB$ (*Prin.* 27); $\triangle AOD \cong \triangle COB$ (*s.a.a. = s.a.a.*).

3. $\angle A = \angle B$ (Given); $CA = CB$ (§ 120); $CD = CF$ (Given); $DA = FB$ (*Prin.* 6); $AE = EB$ (Given); $\triangle DEA \cong \triangle FEB$ (*s.a.s. = s.a.s.*).

4. $AC = AB$, $A'C' = A'B'$, $\angle BCD = \frac{1}{2}\angle C$, and $\angle B'C'D' = \frac{1}{2}\angle C'$ (Given). $\angle C = \angle C'$ (Given); $\angle C = \angle B$, $\angle C' = \angle B'$ (§ 80); $\angle C = \angle B = \angle C' = \angle B'$ (Sub.); $\angle BCD = \angle B'C'D'$ (*Prin.* 8); $CD = C'D'$ (Given); $\triangle CDB \cong \triangle C'D'B'$ (*s.a.a. = s.a.a.*); $CB = C'B'$ (*Prin.* 38); $\triangle ABC \cong \triangle A'B'C'$ (*a.s.a. = a.s.a.*).



5. $AC = AB$, $OC = OB$ (Given); $\angle COD = \angle DOB$ (§ 83); $OD = OD$ (Iden.); $\triangle ODC \cong \triangle ODB$ (*s.a.s. = s.a.s.*).

6. $OD = OD$ (Iden.); $\angle ODC = \angle ODB$ = rt. $\angle$ (Given and *Prin.* 26); $CD = DB$ (Given); $\triangle ODC \cong \triangle ODB$ (*s.a.s.* = *s.a.s.*); $OC = OB$ (*Prin.* 38); in like manner $AC = AB$; $AO = AO$ (Iden.); $\triangle AOC \cong \triangle AOB$ (*s.s.s.* = *s.s.s.*).

7. The triangles are congruent by *hy.l.* = *hy.l.* or by *l.l.* = *l.l.*

8. $\angle 1 = \angle 2$ (Given); $AC \perp BD$ (§ 26); $AO = OC$ and $DO = OB$ (Given); $\triangle AOB \cong \triangle BOC \cong \triangle COD \cong \triangle DOA$ (*l.l.* = *l.l.*); $AB = BC = CD = DA$ (*Prin.* 38).

9. The right triangles formed are congruent by *l.l.* = *l.l.*; the lines are equal by *Prin.* 38.

10. rt. $\triangle ADE \cong$ rt. $\triangle OST$ (*hy.l.* = *hy.l.*); $DE = ST$ (*Prin.* 38); rt. $\triangle ADB \cong$ rt. $\triangle OSP$ (*hy.l.* = *hy.l.*); $DB = SP$ (*Prin.* 38); $EB = TP$ (*Prin.* 6); $CB = RP$ (*Prin.* 7); $\angle B = \angle P$ (*Prin.* 38); $AB = OP$ (Given); $\triangle ABC \cong \triangle OPR$ (*s.a.s.* = *s.a.s.*).

11. The corresponding change in the proof makes rt. $\triangle ADB \cong$ rt. $\triangle OSP$ by *l.a.* = *l.a.*; then finish as in Ex. 10.

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Suggestion: The last of the special cases of congruence for right triangles is given on p. 79. The first seven exercises, which involve altitudes, on pp. 83 and 84 afford an opportunity for the use of the four cases of congruence for right triangles, summarized in § 128. From this point on pupils should be encouraged to use the special cases for right triangles whenever possible since they are generally easier to apply than the combinations of § 127.

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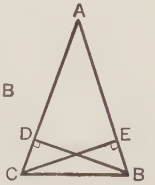
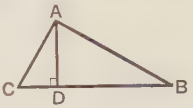
Exercises. Perpendiculars, Medians, and Angle Bisectors

1. $AB = AC$ (Given); $\angle BDA = \angle ADC$ = rt. $\angle$ (§ 133); $AD = AD$ (Iden.); rt. $\triangle ADB \cong$ rt. $\triangle ADC$ (*hy.l.* = *hy.l.*); $CD = DB$ (*Prin.* 38).

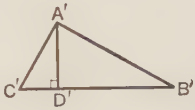
2. $AB = AC$ (Given); $\angle B = \angle C$ (§ 80); $\angle CEB = \angle CDB$ = rt. $\angle$ (Given); $CB = CB$ (Iden.); rt. $\triangle CEB \cong$ rt. $\triangle BDC$ (*hy.a.* = *hy.a.*); $CE = BD$ (*Prin.* 38).

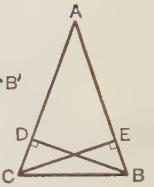


3. $AB = AC$ (Given); $\frac{1}{2}AB = \frac{1}{2}AC$ (*Prin.* 8); $\frac{1}{2}AB = EB$, $\frac{1}{2}AC = CD$ (Given); $EB = DC$ (Sub.); $\angle B = \angle C$ (§ 80); rt. $\triangle EFB \cong$ rt. $\triangle DGC$ (*hy.a.* = *hy.a.*); $EF = DG$ (*Prin.* 38).



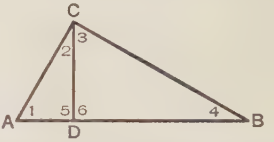
4. $\triangle ABC \cong \triangle A'B'C'$ (Given); $AC = A'C'$, $\angle C = \angle C'$ (*Prin.* 38); $\angle ADC = \angle A'D'C' =$ rt. $\angle$ (§ 133); rt. $\triangle ADC \cong$ rt. $\triangle A'D'C'$ (*hy.a.* = *hy.a.*); $AD = A'D'$ (*Prin.* 38).





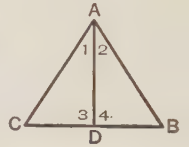
5. $CE = BD$ (Given); $\angle CEB = \angle CDB =$ rt. $\angle$ (§ 133); $CB = CB$ (Iden.); rt. $\triangle CEB \cong$ rt. $\triangle BDC$ (*hy.l.* = *hy.l.*); $\angle B = \angle C$ (*Prin.* 38); $AB = AC$ (§ 120).

6. $\angle 1 + \angle 2 = 90°$ (§ 108); $\angle 3 + \angle 2 = 90°$ (Given); $\angle 1 = \angle 3$ (*Prin.* 28). In like manner, $\angle 2 = \angle 4$, $\angle 5 = \angle 6 =$ rt. $\angle$ (§ 133 and *Prin.* 26).

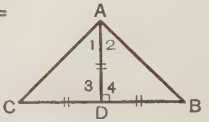


7. $\triangle ACE \cong \triangle ABD$ (*s.a.s.* = *s.a.s.*); $AE = AD$ (*Prin.* 38); $\angle DOA = \angle AOE =$ rt. $\angle$ (§ 133); $AO = AO$ (Iden.); rt. $\triangle AOD \cong$ rt. $\triangle AOE$ (*hy.l.* = *hy.l.*); $\angle OAD = \angle EAO$ (*Prin.* 38).

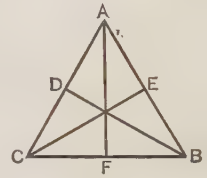
8. $\triangle ADB \cong \triangle ADC$ (*s.a.s.* = *s.a.s.*); $\angle BDA = \angle ADC$ (*Prin.* 38); $AD \perp CB$ (§ 26).



9. $\triangle ADB \cong \triangle ADC$ (*s.s.s.* = *s.s.s.*); $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (*Prin.* 38); $AD \perp CB$ (§ 26). By proof of Ex. 6, $\angle 1 = \angle B$. $\angle 1 = \angle 2 = \angle B$ (Sub.); $DA = DB = \frac{1}{2}BC$ (§ 120).



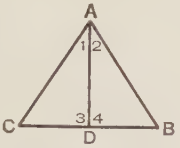
10. In the figure of Ex. 9, $AD = DB$ (Given); $\angle B = \angle 2$ (§ 80). In like manner, $\angle C = \angle 1$. $\angle C + \angle 1 + \angle 2 + \angle B = 180°$ (§ 106); $2\angle 1 + 2\angle 2 = 180°$ (Sub.); $\angle 1 + \angle 2 = 90°$ (*Prin.* 8). $\triangle CAB$ is a rt. $\triangle$ (§ 61) but not necessarily isosceles.

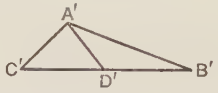


11. $\triangle AFB \cong \triangle CEA \cong \triangle BDC$ (*s.a.s.* = *s.a.s.*); $AF = CE = BD$ (*Prin.* 38).

12. $\triangle ABC \cong \triangle A'B'C'$ (Given); $\triangle ABD \cong \triangle A'B'D'$ (*s.a.s.* = *s.a.s.*); $AD = A'D'$ (*Prin.* 38).

13. rt. $\triangle ADB \cong$ rt. $\triangle ADC$ (*l.a.* = *l.a.*); $AB = AC$ (*Prin.* 38).





14. In the figures of Ex. 15, $\triangle ADB \cong \triangle A'D'B'$ (*a.s.a.* = *a.s.a.*); $AD = A'D'$ (*Prin.* 38).

15. $\angle B = \angle B'$ (§ 111); $\angle DAB = \angle D'A'B'$ (*Prin.* 8); $\triangle ADB \cong \triangle A'D'B'$ (*s.a.a.* = *s.a.a.*).

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Note: Before taking up Propositions 15 and 16, the Oral Exercises on p. 84 should be developed and attention should be called to the significance of the arrows in the figures. The arrows need not be demanded of pupils in their solutions of exercises.

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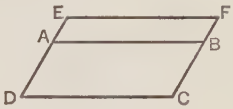
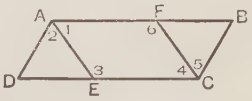
Exercises. Parallelograms



1. *DCFE* or *ABFE* is a parallelogram (§ 152, *a*).



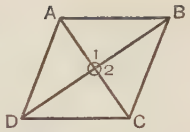
2. *ABCD* is a $\square$ (Given); $\angle A = \angle C$ (§ 145); $\angle 1 = \angle 4$ (*Prin.* 8); $\angle 3 = \angle 6$ (*Prin.* 28); *AFCE* is a $\square$ (152, *c*); $AE = CF$ (§ 147); $AE \parallel CF$ (§ 138).

3. $AE \perp DC$, $CF \perp AB$ (Given); $AB \parallel DC$ (Given); $AE \perp AB$ (§ 99); $AE \parallel CF$ (*Prin.* 17); *AFCE* is a $\square$ (§ 138); $AE = CF$ (§ 147).

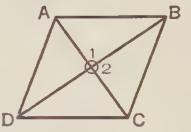
4. *ABCD* is a $\square$ and $\angle C$ = rt. $\angle$ (Given and § 140); $BC = AD$ (§ 147); $\angle D$ = rt. $\angle$ (§ 145); $DC = DC$ (Iden.); rt. $\triangle BCD \cong$ rt. $\triangle ADC$ (*l.l.* = *l.l.*); $BD = AC$ (*Prin.* 38).

Note: This is an important exercise.

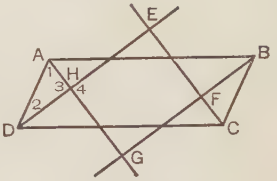
5. $ABCD$ is a $\square$ and $BA = BC$ (Given and § 139); $AO = OC$ (§ 149); $BO = BO$ (Iden.); $\triangle BOA \cong \triangle BOC$ (*s.s.s.* = *s.s.s.*); $\angle 1 = \angle 2$ (*Prin.* 38); $BO \perp AC$ (§ 26).



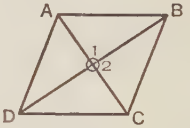
6. $\angle 1 = \angle 2 =$ rt. $\angle$ (Given); $AO = OC$ (§ 149); $BO = BO$ (Iden.); $\triangle BOA \cong \triangle BOC$ (*s.a.s.* = *s.a.s.*); $BA = BC$ (*Prin.* 38); similarly, $BC = DC$; $BC = AD$ (§ 147); $BA = BC = DC = AD$ (Sub.).



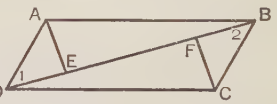
7. $AG \parallel CE$, $DE \parallel BG$ (Ex. 2); $HEFG$ is a $\square$ (§ 138); $\angle DAB + \angle CDA = 180°$ (§ 145); $\angle 1 + \angle 2 = 90°$ (*Prin.* 8); $\angle 1 + \angle 2 + \angle 3 = 180°$ (§ 106); $\angle 3 = 90°$ (*Prin.* 6); $\angle 3 = \angle 4 = 90°$ (*Prin.* 27). $HEFG$ is a rect. (§ 140).



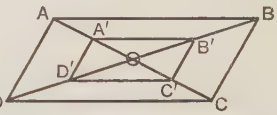
8. $ABCD$ is a $\square$ and $AD = DC$ (Given); $AD = BC$, $DC = AB$ (§ 147); $AD = DC = BC = AB$ (Sub.); $AO = OC$, $BO = OD$ (§ 149); $\triangle AOD \cong \triangle DOC \cong \triangle BOC \cong \triangle AOB$ (*s.s.s.* = *s.s.s.*).



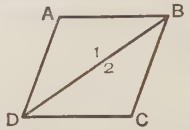
9. $AE \perp BD$, $CF \perp BD$ (Given); $AE \parallel CF$ (*Prin.* 17); $\angle 1 = \angle 2$ (§ 101); $AD = BC$ (§ 147); rt. $\triangle AED \cong$ rt. $\triangle CFB$ (*hy.a.* = *hy.a.*); $AE = CF$ (*Prin.* 38).



10. $BO = OD$, $AO = OC$ (§ 149); $\frac{1}{2} BO = \frac{1}{2} OD$, $\frac{1}{2} AO = \frac{1}{2} OC$ (*Prin.* 8); $\frac{1}{2} BO = B'O$, $\frac{1}{2} OD = OD'$, $\frac{1}{2} AO = A'O$, $\frac{1}{2} OC = OC'$ (Given); $B'O = OD'$, $A'O = OC'$ (Sub.); $A'B'C'D'$ is a $\square$ (§ 152, *b*).

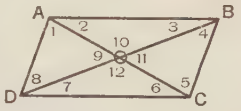


11. $\triangle ABD \cong \triangle CBD$ (*s.a.s.* = *s.a.s.*); $AD = DC$ (*Prin.* 38); $AD = BC$, $DC = AB$ (§ 147); $AD = DC = BC = AB$ (Sub.).



12. Use figure of Ex. 4; $\angle A = 90°$ (Given); $\angle A = \angle C = 90°$ (Sub.); $\angle A + \angle B = 180°$ (§ 145); $\angle B = 90°$ (*Prin.* 6); $\angle B = \angle D = 90°$ (Sub.).

13. (*a*) $AB = DC$, $BC = AD$, $AO = OC$, $BO = OD$. (*b*) $\angle A = \angle C$, $\angle B = \angle D$, $\angle 9 = \angle 11$, $\angle 10 = \angle 12$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$, $\angle 1 = \angle 5$, $\angle 2 = \angle 6$. (*c*) $\triangle AOB \cong \triangle COD$, $\triangle BOC \cong \triangle DOA$, $\triangle BCD \cong \triangle ADB$, $\triangle ADC \cong \triangle ABC$. (*d*) $\angle A$ and $\angle D$, $\angle A$ and $\angle B$, $\angle C$ and $\angle D$, $\angle C$ and $\angle B$, $\angle 9$ and $\angle 10$, $\angle 9$ and $\angle 12$, $\angle 11$ and $\angle 10$, $\angle 11$ and $\angle 12$.



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Note: Proposition 24 and some of the exercises on p. 95 are not of prime importance and may be omitted or reserved for future study if time permits. The teacher should constantly bear in mind that not all the material in this text is for intensive study. Selections must be made of propositions and of exercises. See that the class keeps to scheduled time. All of the most important parts of the work must be thoroughly covered; whatever time remains may be given to the work of secondary importance.

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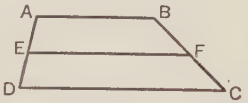
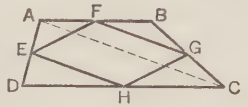
Exercises. Trapezoids

1. $EF \parallel AB$, $EF \parallel DC$ (§§ 157, 156); $ABFE$ and $EFCD$ are trapezoids (§ 137).



2. In the figure of Ex. 1, $AB \parallel DC$ (Given); $\angle A + \angle D = 180°$, $\angle B + \angle C = 180°$ (§ 104).

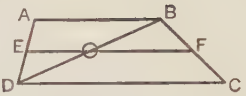
3. $FG \parallel AC$, $FG = \frac{1}{2}\ AC$, $EH \parallel AC$, $EH = \frac{1}{2}\ AC$ (§ 155); $FG \parallel EH$ (*Prin.* 16); $FG = EH$ (Sub.); $FGHE$ is a $\square$ (§ 152, *a*).



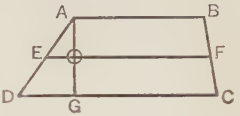
4. Draw AE and BF each $\perp DC$ (*Prin.* 15); $AE \parallel BF$ (*Prin.* 17); $AE = BF$ (§ 148); $AD = BC$ (Given); rt. $\triangle AED \cong$ rt. $\triangle BFC$ (*hy.l.* = *hy.l.*); $\angle D = \angle C$ (*Prin.* 38); $\angle A = \angle B$ (*Prin.* 28).

5. $\angle D = \angle C$ (Ex. 4); $DC = DC$ (Iden.); $AD = BC$ (Given); $\triangle ADC \cong \triangle BCD$ (*s.a.s.* = *s.a.s.*). Also $\triangle DAB \cong \triangle CBA$ (*s.a.s.* = *s.a.s.*).

6. Use proof of Ex. 5 and $AC = BD$ (*Prin.* 38).

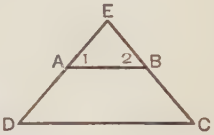


7. $EF \parallel DC$ (§ 156); $BO = OD$ (§ 154). If O' is the common point of AC and EF, similarly $AO' = O'C$.



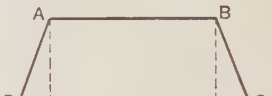
8. Let AG be an altitude of $ABCD$. $EF \parallel DC$ (§ 156); $AO = OG$ (§ 154).

9. $\angle D = \angle C$ by proof of Ex. 4; $\angle 1 = \angle D$, $\angle 2 = \angle C$ (§ 102); $\angle 1 = \angle 2$ (Sub.); $ED = EC$, $EA = EB$ (§ 120).



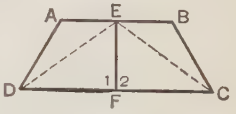
10. In trapezoid $ABCD$ if $\angle D = \angle C$, $\angle A = \angle B$ (*Prin.* 28).

11. Draw AE and BF each $\perp DC$ (*Prin.* 15); $AE \parallel BF$ (*Prin.* 17); $AE = BF$ (§ 148); $\angle D = \angle C$ (Given); rt. $\triangle AED \cong$ rt. $\triangle BFC$ (*a.l.* = *a.l.*, § 113); $AD = BC$ (*Prin.* 38).



12. In the figure of Ex. 4, rt. $\triangle AEC \cong$ rt. $\triangle BFD$ (*hy.l.* = *hy.l.*); $\triangle ADC \cong \triangle BCD$ (*s.a.s.* = *s.a.s.*); $AD = BC$ (*Prin.* 38).

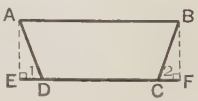
13. Draw ED and EC (*Prin.* 13); $\angle A = \angle B$ as in Ex. 4; $AE = EB$, $AD = BC$ (Given); $\triangle ADE \cong \triangle BCE$ (*s.a.s.* = *s.a.s.*); $ED = EC$ (*Prin.* 38); $\triangle EFD \cong \triangle EFC$ (*s.s.s.* = *s.s.s.*); $\angle 1 = \angle 2$ (*Prin.* 38); $EF \perp DC$ (§ 26); $EF \perp AB$ (§ 99).



14.

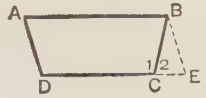
First Figure.

$\angle 1 = \angle 2$ (*Prin.* 28); $\angle DEA = \angle BFC =$ rt. $\angle$ (Cons.); $AD = BC$ (Given); $\triangle AED \cong \triangle BFC$ (*hy.a.* = *hy.a.*); $AE = BF$ (*Prin.* 38); $AE \parallel BF$ (*Prin.* 17); $ABFE$ is a $\square$ (§ 152, *a*); $AB \parallel DC$ (§ 138).



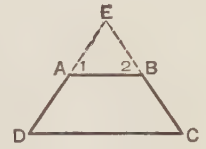
Second Figure.

$BE \parallel AD$ (Cons.); $\angle D + \angle E = 180°$ (§ 104); $\angle 1 + \angle 2 = 180°$ (*Prin.* 29); $\angle D = \angle 1$ (Given); $\angle 2 = \angle E$ (*Prin.* 28); $BC = BE$ (§ 120); $BC = AD$ (Given); $BE = AD$ (Sub.); $ABED$ is a $\square$ (§ 152, *a*); $AB \parallel DC$ (§ 138).



Third Figure.

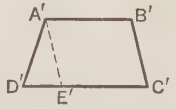
$\angle D = \angle C$ (Given); $ED = EC$ (§ 120); $AD = BC$ (Given); $AE = EB$ (*Prin.* 6); $\angle 1 = \angle 2$ (§ 80); $\angle D + \angle C = 180° - \angle E$, $\angle 1 + \angle 2 = 180° - \angle E$ (§ 107); $\angle D + \angle C = \angle 1 = \angle 2$ and $2\ \angle D = 2\ \angle 1$ (Sub.); $\angle D = \angle 1$ (*Prin.* 8); $AB \parallel DC$ (§ 116).



Note on Ex. 14: No definite directions are given for drawing the dotted lines in order to give pupils practice in interpreting constructions, which is a proper part of their training.

15. *ABFD* and *ABCE* are ▱ (§ 152, *a*); $AD = BF$, $AE = BC$ (§ 147); $DF = EC$ (Given); $EF = EF$ (Iden.); $DE = FC$ (*Prin.* 6); $\triangle ADE \cong \triangle BFC$ (*s.s.s.* = *s.s.s.*).

16. Draw $AE \parallel BC$ and $A'E' \parallel B'C'$ (*Prin.* 14); $AB = EC$, $A'B' = E'C'$ (§ 147); $DE = D'E'$ (*Prin.* 6); $\triangle ADE \cong \triangle A'D'E'$ (*s.s.s.* = *s.s.s.*); $\angle D = \angle D'$ (*Prin.* 38); in like manner $\angle C = \angle C'$. Apply the Method of Superposition to complete the proof.



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Note: The teacher may give a few simple oral exercises of a numerical character to emphasize the Facts of §§ 158 and 159. There are numerous places throughout the text where this should be done, especially in those cases which lend themselves to easy numerical computations. In many such instances, as in § 158, the conclusion is a formula and may be so treated.

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Suggestion: In exercises consisting of a number of similar parts, such as Exs. 6, 7, 8, 9, and 11, only a few of the parts need be assigned for one lesson. The remaining parts may be distributed over subsequent lessons.

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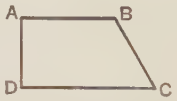
Exercises. Angles of Polygons

1. $n = 4$, $\frac{(n-2)}{n}\,180° = 90°$ (§ 167).

2. $\angle A + \angle B + \angle C + \angle D = 360°$ (§ 164); $\angle A + \angle D = 180°$ (Given and *Prin.* 5); $\angle B + \angle C = 180°$ (*Prin.* 6).



3. $\angle A = \angle C$, $\angle B = \angle D$ (Given); $\angle A + \angle B + \angle C + \angle D = 360°$ (§ 164); $2\ \angle A + 2\ \angle D = 360°$, $2\ \angle A + 2\ \angle B = 360°$ (Sub.); $\angle A + \angle D = 180°$, $\angle A + \angle B = 180°$ (*Prin.* 8); $AB \parallel DC$, $AD \parallel BC$ (§ 117); $ABCD$ is a $\square$ (Def.).

4. Use § 164. 360°, 540°, 720°, 1,080°, 1,440°, 8,640°, 18,000°.

5. Use § 167. 90°, 72°, 60°, 45°, 36°, $7\frac{1}{2}°$, $3\frac{9}{17}°$.

6. $(n-2)\ 180° = 720°$ (Given and § 164); $n - 2 = 4$; $n = 6$. The remaining answers are 7, 9, 13, and 22 sides.

7. $2\ (n-2)$ rt. $\measuredangle = 8$ rt. $\measuredangle$ (Given and § 164); $2\ (n-2) = 8$; $n = 6$. The remaining answers are 16, 6, 12, $\frac{1}{2}\ (n+4)$, and $(n+2)$ sides.

8. $\frac{360°}{n} = 72°$ (Given and § 167); $\frac{5}{n} = 1$; $n = 5$. The remaining answers are 10, 18, 20, 40, 60, 27, 35, and 75 sides.

9. At any vertex an interior angle + its exterior angle = 180° (*Prin.* 29); 180° − 108° = 72°, each exterior angle. The answers are 5, 8, 10, 12, 15, 18, 16, 24, 40, 42, and 52 sides.

10. Each angle of the regular pentagon equals 108° (§ 167); an adjacent angle = 72°; each angle at a point of the pentagram equal 36°; their sum equals 180°.

11. By a solution similar to that of Ex. 10, the answers are 360° and 720°.

12. The sum of the five given angles = 540° (§ 164); the sum of the four given angles = 395° (*Prin.* 5); the remaining angle = 145° (*Prin.* 6).

13. Assume the angles to be the successive angles of the irregular pentagon. Angles of 90° and 85° (those adjacent to 90° and 95°) are two angles of a triangle of which the third angle is 5°. This is the angle at one point. The other angles at successive points are 10°, 30°, 75°, and 60°. The sum of the angles at the points of the irregular pentagram is 5° + 10° + 30° + 75° + 60° = 180°.

14. $\frac{360°}{n} = 7°$ (Given and § 167) $n = \frac{360}{7} = 51\frac{3}{7}$ sides, which is impossible.

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Suggestion: The teacher should remember that the Minimum Course for Chapter One ends with § 181. Avoid giving so much time to one chapter that the work in the other chapters is hurried. **Make selections from the Supplementary Exercises and the Miscellaneous Exercises accordingly.** However, do not overlook for review and diagnostic purposes the importance of the Summary and the Review Oral Exercises in each chapter. Both of the latter lend themselves admirably to a rapid-fire oral discussion and the Summary may be used for a comprehensive written test.

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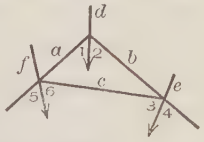
Exercises. Concurrent Lines

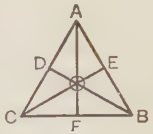
1. Since an exterior angle of a triangle can not equal a st. ∠ (§§ 110, 106), two exterior angles can not equal 360° and ∠6 + ∠3 ≠ 180°. Lines *f* and *e* intersect (at some point *O*) (*Prin.* 18); ∠5 = ∠6 (Given); *O* is equidistant from *a* and *c* (§ 172); ∠3 = ∠4 (Given); *O* is equidistant from *c* and *b* (§ 172); that is, *O* is equidistant from *a* and *b* and lies on *d* (§ 173); *d*, *e*, and *f* are concurrent (§ 174).



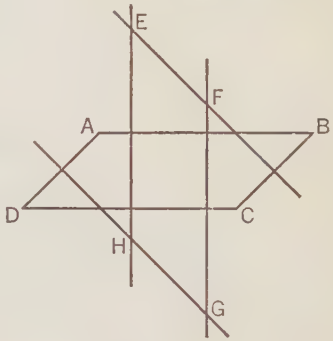
2. Let *O* be the mid-point of *AB* (*Prin.* 12); *AO* = *OB* (§ 37); produce *CO* to *D* making *OD* = *CO* (*Prin.* 11); *ADBC* is a □ (§ 152, *b*); ∠*C* = rt. ∠ (Given); *ADBC* is a rectangle

(§ 140); $AB = CD$ (by proof of Ex. 4, p. 91); $AO = OC$ (*Prin.* 8); $OA = OC = OB$. That is, O is equidistant from A, B, and C and is the circumcenter (§ 176 and Note).

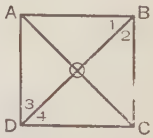
3. The angle bisectors, the medians, and the altitudes of an equilateral triangle are the same three lines. CE and BD intersect AF at O, so that $AO = 2\,OF$ (§§ 177, 181, and 178).



4. For rectangles only, otherwise they form a parallelogram. $AB \parallel DC$, $AD \parallel BC$ (Given); $EH \perp DC$, $FG \perp AB$ (Given); $FG \perp DC$ (§ 99); $EH \parallel FG$ (§ 98). In like manner, $EF \parallel HG$. $EFGH$ is a $\square$ (§ 138).



5. The bisectors of the angles of a square or rhombus are concurrent. $ABCD$ is a square (or rhombus); BD forms two congruent triangles (*s.a.s.* = *s.a.s.*); $\angle 1 = \angle 2 = \angle 3 = \angle 4$ (*Prin.* 38 and Sub.). In like manner, AC bisects $\angle A$ and $\angle C$. AC and BD intersect at O (§ 149), that is AO, BO, CO, and DO are concurrent (§ 174).



6. CD and AE are medians (Given); $AO = 2\,OE$, $OC = 2\,OD$ (§ 181); $OE = 4$, $OC = 9$ (Given); $AO = 8$, $OD = 4\frac{1}{2}$ (Sub.).

7. CD and AE are medians intersecting at O (Given); BOP is a median (§ 179 and *Prin.* 13); $PA = \frac{1}{2}AC$ (§ 132); $AC = AE$ (Given); $PA = \frac{1}{2}AE$ (Sub.).

8. Since the altitude of an isos. $\triangle$ is the bisector of the vertex angle (*hy.l.* = *hy.l.* and *Prin.* 38). a, b, and c are concurrent (§ 177).

9. $DE \parallel FB$ and $DF \parallel EB$ (§ 155); $DEBF$ is a $\square$ (§ 138); $DG = GB = \frac{1}{2}DB$ (§ 149).

Note: Ex. 9 is suitable as an "extra credit" exercise since other conclusions are possible such as, $OG:\ DO:\ GB = 1:\ 2:\ 3$.

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Supplementary Exercises. Angles of Triangles

1. $\frac{1}{2}\angle C + \frac{1}{2}\angle B = 180° - \angle P = 80°$ (§ 107 and Sub.); $\angle C + \angle B = 160°$ (*Prin.* 7); $\angle A + \angle B + \angle C = 180°$ (§ 106); $\angle A = 20°$ (*Prin.* 6). It is interesting also to develop the formula, $A = 180 - 2\ (180 - P) = 2\,P - 180° = 2\,(P - 90°)$. The remaining answers are 60°, 45°, 33°, and $2\,(m - 90°)$.

2. $\angle A + \angle B = 90°$ (§ 108); $x + 2\,x = 90$; $3\,x = 90$; $x = 30$; $2\,x = 60$. The answers are

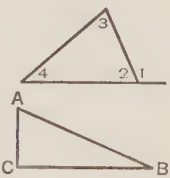
	a	b	c	d	e	f	g
$\angle A$	30	15	60	36	35	60	$64\frac{2}{7}$
$\angle B$	60	75	30	54	55	30	$25\frac{5}{7}$

3. $\angle A = 70°$ (Given); $\angle A + \angle 1 = 90°$ (§ 108); $\angle 2 + \angle 1 = 90°$ (§ 108); $\angle A = \angle 2$ (*Prin.* 28); $\angle 2 + \angle 3 = 180°$ (*Prin.* 29); $\angle A = 70° = \angle 2$ (Given and Sub.); $\angle 3 = 110°$ (*Prin.* 6).

Note: It might be interesting and profitable to ask pupils, "What is the relation of $\angle A$ to $\angle 3$?" "Is this always true?" If so, prove that "the vertex angle of an isosceles triangle and the angle included between the altitudes drawn from the extremities of the base to the opposite sides are supplementary."

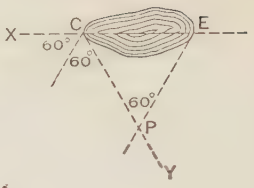
4. $\angle 1 = \angle 4 + \angle 3$ (§ 110); $\angle 4 = \angle 4$ (Iden.); $\angle 1 - \angle 4 = \angle 3$ (*Prin.* 6).

5. $\angle A + \angle B + \angle C = 180°$ (§ 106); $\angle A + \angle B = \angle C$ (Given); $\angle C + \angle C = 180°$ (Sub.); $\angle C = 90°$ (*Prin.* 8); $\angle ACB$ is a rt. $\triangle$ (Def.).

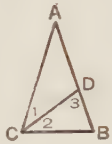


6. The only instrument needed is a piece of cardboard or wood cut to an angle of 60°. Place stakes at E and C. Determine and mark CX. Turn an angle of 120° from XC. Mark CY. Find the point

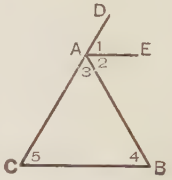
on CY (such as P) where one side of the $60°$ angle rests on CY and the other side produced passes through E. Drive a stake in at P. The distance between P and C is also the distance between C and E. $\angle C = \angle P = \angle E = 60°$ and $PC = PE = CE$.



7. $\angle A = 36°$ (Given); $\angle C + \angle B = 144°$ (§ 107); $\angle C = \angle B$ (§ 80); $2\ \angle C = 144°$ (Sub.); $\angle C = 72°$ (*Prin.* 8); $\angle 1 = \angle 2 = \frac{1}{2}\angle C$ (Given); $\angle 1 = \angle 2 = 36°$ (Sub.).



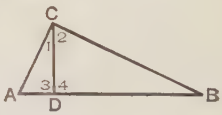
8. $\angle 1 = \angle 2$ (Cons.); $\angle 4 = \angle 5$ (§ 80); $\angle 1 + \angle 2 = \angle 4 + \angle 5$ (§ 110); $2\ \angle 1 = 2\ \angle 4 = 2\ \angle 5$ (Sub.); $\angle 1 = \angle 4 = \angle 5 = \frac{1}{2}(\angle 1 + \angle 2)$ (*Prin.* 8).



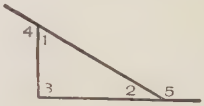
9. Case I. When the vertex angle is $60°$. $\angle A = 60°$ (Given); $\angle B + \angle C = 120°$ (§ 107); $\angle B = \angle C$ (§ 80); $2\ \angle B = 2\ \angle C = 120°$ (Sub.); $\angle B = \angle C = 60° = \angle A$ (Sub.).

Case II. When a base angle is $60°$. $\angle B = 60°$ (Given); $\angle B = \angle C = 60°$ (§ 80 and Sub.); $120° = 180° - \angle A$ (Sub. and § 107) or $\angle A = 60°$.

10. $\angle 3 = \angle 4 = 90°$ (Given); $\angle 1 + \angle 2 = 90°$ (Given); $\angle 1 + \angle A = 90°$ (§ 108); $\angle 2 + \angle B = 90°$ (§ 108); $\angle 2 = \angle A$, $\angle 1 = \angle B$ (*Prin.* 28).



11. $\angle 4 + \angle 1 + \angle 5 + \angle 2 = 360°$ (*Prins.* 29 and 5); $\angle 4 + \angle 5 = 270°$ (Given); $\angle 1 + \angle 2 = 90°$ (*Prin.* 6); $90° = 180° - \angle 3$ (Sub. and § 107) or $\angle 3 = 90°$.



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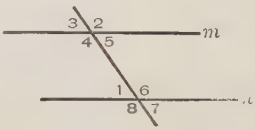
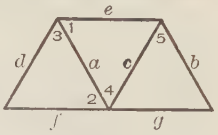
Supplementary Exercises. Parallel Lines

12. $m \parallel n$, $\angle 1 = 40°$ (Given); $\angle 6 = 140°$ (*Prin.* 29, Sub.); $\angle 5 = \angle 1 = 40°$, $\angle 6 = \angle 4 = 140°$ (§ 101, Sub.); $\angle 3 = 40°$, $\angle 7 = 40°$, $\angle 2 = 140°$, $\angle 8 = 140°$ (*Prin.* 27, Sub.).

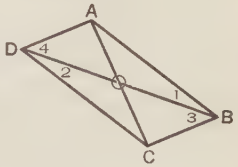


13. Let $4x = \angle 1$ and $5x = \angle 2$. $\angle 2 = \angle 6$ (§ 102); $\angle 1 + \angle 6 = 180°$ (*Prin.* 29); $\angle 1 + \angle 2 = 180°$ (Sub.); $4x + 5x = 180°$ (Sub.); $x = 20°$, $\angle 1 = 80°$, $\angle 2 = 100°$. The remaining angles are either 80° or 100°.



14. The angles 1, 2, 3, 4, 5 are each 60°. $e \parallel f$, $c \parallel d$, and $a \parallel b$ (§ 114).

15. $\triangle AOB \cong \triangle COD$, $\triangle AOD \cong \triangle COB$ (*s.a.s.* = *s.a.s.*); $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (*Prin.* 38); $AB \parallel DC$, $BC \parallel AD$ (§ 114).



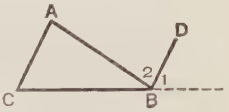
16. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (Given); $\angle 1 + \angle 3 = \angle 2 + \angle 4$ (*Prin.* 5); $m \parallel n$ (§ 114).

17. $\angle 1 = \angle 3 = \angle 4$ (§§ 102, 80, Sub.).

18. $\triangle BAC \cong \triangle DCA$ (*s.a.s.* = *s.a.s.*); $\angle DAC = \angle ACB$ (*Prin.* 38); $AD \parallel BC$ (§ 114).

19. This is a restatement of § 98.

20. $\angle 1 + \angle 2 + \angle B = 180°$ (§ 54); $\angle 1 = \angle C$ (§ 102); $\angle 2 = \angle A$ (§ 101); $\angle A + \angle B + \angle C = 180°$ (Sub.).



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Supplementary Exercises. Congruent Triangles

21. $AB = AC$ (Given); $HB = GC$ (*Prin.* 8); $\angle B = \angle C$ (§ 80); $BP = CP$ (Given); $\triangle HBP \cong \triangle GPC$ (*s.a.s.* = *s.a.s.*).

22. $\triangle BAC \cong \triangle DAC$ (*s.a.s.* = *s.a.s.*); $BA = DA$, $\angle BAO = \angle DAO$ (*Prin.* 38); $\triangle ABO \cong \triangle AOD$ (*s.a.s.* = *s.a.s.*).

23. Let AC produced and BD intersect at E. rt. $\triangle CEB \cong$ rt. $\triangle CED$ (*l.l.* = *l.l.*); $BC = DC$; $\angle BCE = \angle ECD$ (*Prin.* 38); $\angle ACB = \angle ACD$ (*Prin.* 28); $\triangle ABC \cong \triangle ACD$ (*s.a.s.* = *s.a.s.*).

24. $\triangle AOC \cong \triangle AOB$ (*s.a.s.* = *s.a.s.*); $AC = AB$, $\angle CAD = \angle BAD$ (*Prin.* 38); $\triangle ADC \cong \triangle ADB$ (*s.a.s.* = *s.a.s.*).

25. rt. $\triangle ADC \cong$ rt. $\triangle PTS$ (*hy.l.* = *hy.l.*); $\angle C = \angle S$ (*Prin.* 38); $\angle ABC \cong \triangle PRS$ (*s.a.s.* = *s.a.s.*).

26. rt. $\triangle ADC \cong$ rt. $\triangle PTS$ (*hy.l.* = *hy.l.*); $\angle ADB = \angle PTR$ = rt. $\angle$ (*Prin.* 28); rt. $\triangle ADB \cong$ rt. $\triangle PTR$ (*hy.l.* = *hy.l.*); $CD + DB = ST + TR$ (*Prins.* 38 and 5); $\triangle ABC \cong \triangle PRS$ (*s.s.s.* = *s.s.s.*).

27. rt. $\triangle ADC \cong$ rt. $\triangle PTS$ (*hy.l.* = *hy.l.*); rt. $\triangle ADB \cong$ rt. $\triangle PTR$ (*a.l.* = *a.l.*); $CD + DB = ST + TR$ (*Prins.* 38 and 5); $\triangle ABC \cong \triangle PRS$ (*s.s.s.* = *s.s.s.*).

28. rt. $\triangle ADC \cong$ rt. $\triangle PTS$, rt. $\triangle ADB \cong$ rt. $\triangle PTR$ (*a.l.* = *a.l.*); $\triangle ABC \cong \triangle PRS$ (*a.s.a.* = *a.s.a.*).

29. rt. $\triangle ADC \cong$ rt. $\triangle PTS$ (*a.l.* = *a.l.*); $AC = PS$ (*Prin.* 38); $\triangle ABC \cong \triangle PRS$ (*a.s.a.* = *a.s.a.*).

30. rt. $\triangle ADE \cong$ rt. $\triangle OST$ (*hy.l.* = *hy.l.*); rt. $\triangle ADB \cong$ rt. $\triangle OSP$ (*hy.l.* = *hy.l.*); $EB = TP$ (*Prin.* 6); $CB = RP$ (*Prin.* 7); $\angle B = \angle P$ (*Prin.* 38); $\triangle ABC \cong \triangle OPR$ (*s.a.s.* = *s.a.s.*).

31. rt. $\triangle ADE \cong$ rt. $\triangle OST$ (*hy.l.* = *hy.l.*); rt. $\triangle ADB \cong$ rt. $\triangle OSP$ (*a.l.* = *a.l.*); $AB = OP$ (*Prin.* 38); $EB = TP$ (*Prin.* 6); $CB = RP$ (*Prin.* 7); $\triangle ABC \cong \triangle OPR$ (*s.a.s.* = *s.a.s.*).

32. rt. $\triangle ADE \cong$ rt. $\triangle OST$ (*hy.l.* = *hy.l.*); $\angle AEC = \angle OTR$ (*Prin.* 38); $CE = RT$ (*Prin.* 8); $\triangle AEC \cong \triangle OTR$ (*s.a.s.* = *s.a.s.*); $\angle C = \angle R$, $AC = OR$ (*Prin.* 38); $\triangle ABC \cong \triangle OPR$ (*s.a.s.* = *s.a.s.*).

33. rt. $\triangle ADE \cong$ rt. $\triangle OST$ (*hy.l.* = *hy.l.*); $\angle AEB = \angle OTP$ (*Prin.* 28); $\triangle AEB \cong \triangle OTP$ (*a.s.a.* = *a.s.a.*); $AB = OP$, $\angle B = \angle P$, $EB = TP$ (*Prin.* 38); $CB = RP$ (*Prin.* 7); $\triangle ABC \cong \triangle OPR$ (*s.a.s.* = *s.a.s.*).

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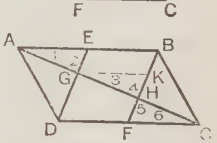
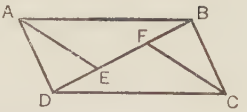
Supplementary Exercises. Parallelograms

35. $AE = FC$ (*Prin.* 8); $AE \parallel FC$ (Given); $AECF$ is a $\square$ (§ 152); $AF = EC$, $AF \parallel EC$ (§§ 147, 138).

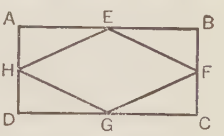
36. Draw $GK \parallel AB$ (*Prin.* 14); $EG \parallel BK$ (Ex. 35); $EBKG$ is a $\square$ (§ 138); $EB = GK$ (§ 147); $AE = GK = FC$ (§ 147, *Prin.* 8, Sub.); $\angle 1 = \angle 3 = \angle 6$ (§§ 102, 101, Sub.); $\angle 2 = \angle 4 = \angle 5$ (§ 102, *Prin.* 27 and Sub.); $\triangle GEA \cong \triangle HKG \cong HFC$ (*s.a.a.* = *s.a.a.*); $AG = GH = HC$ (*Prin.* 38).



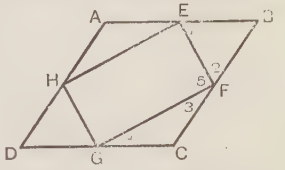
37. $AD = BC$ (§ 147); $\angle A = \angle C$ (§ 145); $\angle DAE = \angle BCF$ (*Prin.* 8); $\angle ADE = \angle CBF$ (§ 101); $\triangle ADE \cong \triangle CBF$ (*a.s.a.* = *a.s.a.*); $AE = CF$ (*Prin.* 38).



38. $ABCD$ is a $\square$ and $\angle A$ = rt. $\angle$ (Given and § 140); $\angle A = \angle C$ = rt. $\angle$ (§ 145 and Sub.); $\angle B = \angle D$ = rt. $\angle$ (§ 145 and Sub.); $AE = EB = DG = GC, AH = HD = BF = FC$ (*Prin.* 8). rt. $\triangle HAE \cong$ rt. $\triangle FBE \cong$ rt. $\triangle FCG \cong$ rt. $\triangle HDG$ (*l.l.* = *l.l.*); $HE = EF = FG = GH$ *Prin.* 38); $HEFG$ is a $\square$ (§ 151); $HEFG$ is a rhombus (§ 139).



39. $ABCD$ is a $\square$ and $AB = AD$ (Given and § 139); $AB = DC = AD = BC$ (§ 147, Sub.); $\triangle HDG \cong \triangle EBF$, $\triangle HAE \cong \triangle FCG$ (*s.a.s.* = *s.a.s.*). $HG = EF, HE = GF$ (*Prin.* 38); $HEFG$ is a $\square$ (§ 151); $\angle 1 + \angle 2 + \angle B + \angle 3 + \angle C + \angle 4 = 360°$ (§ 106, *Prin.* 5); $\angle B + \angle C = 180°$ (§ 145); $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$ (*Prin.* 6); $\angle 1 = \angle 2, \angle 3 = \angle 4$ (§ 80); $\angle 2 + \angle 3 = 90°$ (Sub., *Prin.* 8); $\angle 2 + \angle 5 + \angle 3 = 180°$ (§ 54); $\angle 5 = 90°$ (*Prin.* 6); $HEFG$ is a rectangle (§ 140).



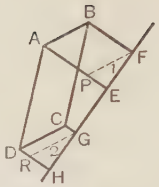
40. $ABCD$ is a $\square$ (Given); $\triangle ADC \cong \triangle BCD$ (*s.s.s.* = *s.s.s.*); $\angle C = \angle D$ (*Prin.* 38); $\angle C + \angle D = 180°$ (§ 145); $2\ \angle C = 180°$ (Sub.); $\angle C = 90°$ (*Prin.* 6); $ABCD$ is a rectangle (§ 140).

41. In the figure of Ex. 40, $ABCD$ is a $\square$ and $\angle A = 90°$ (Given); $\angle A = \angle C = 90°$ (§ 145, Sub.); $\angle B = \angle D = 90°$ (§ 145, Sub.).

42. The proof depends upon showing that $a = b$ and $c = d$. $\angle 1 = \angle 4$ (§ 80); $\angle 4 = \angle 2$ (§ 102); $\angle 1 = \angle 4 = \angle 2$ (Sub.); $a = b$ and similarly $c = d$ (§ 120).

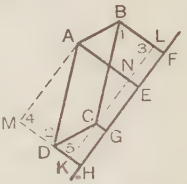
43. Draw $FP \parallel BA$ and $GR \parallel CD$ (*Prin.* 14); $PF = AB = CD = RG$ (§ 147, Sub.); $PF \parallel AB \parallel CD \parallel RG$ (§ 138, *Prin.* 16); $\angle 1$

$= \angle 2$ (§ 102); $\triangle PEF \cong \triangle RHG$ (*hy.a.* = *hy.a.*); $PE = RH$ (*Prin.* 38); $AP = BF$ and $CG = DR$ (§ 147); $CG + AP + PE = DR + BF + RH$ (*Prin.* 5); $CG + AE = DH + BF$ (*Prin.* 4 and Sub.).



43. Optional proof. Through C draw $KL \parallel HF$. The proof depends upon showing that $DK + BL = AN$.

Produce HD making $DM = BL$; $\angle 2 = \angle 1$ (§ 134); $\triangle MDA \cong \triangle LBC$ (*s.a.s.* = *s.a.s.*); $\angle 4 = \angle 3$ = rt. $\angle$ (*Prin.* 38, Sub.); $\angle 5$ = rt. $\angle$ (§ 99); $MANK$ is a $\square$ (§ 138); $MK = AN$ (§ 147); $DK + BL = AN$ (Sub.).



44. The only instrument needed is a piece of cardboard or wood cut to an angle of 90°. Place stakes at D and C and turn a rt. $\angle$ at D and at C from line DC, making $DA = CB$. The distance between A and B is also the distance between C and D, because $DCBA$ is a rectangle.

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Supplementary Exercises. Angles of Polygons

45. If we disregard the two sides of the polygon which meet at the vertex selected, each of the other sides may be taken as the base of a triangle. There are $(n - 2)$ triangles formed. The sum of the angles is $(n - 2)$ st. $\angle$s.

46. Not more than one, four, three, three, and three for the respective polygons. For the triangles, three angles cannot be rt. $\measuredangle$s, because their sum would be greater than 2 rt. $\measuredangle$s; if two angles were right angles, the third angle would be zero. For the quardilateral, four angles may be right angles, because their sum would equal four right angles. For the pentagon, five angles cannot be right angles, because the sum of the angles of a pentagon is $(5 - 2) \cdot 2$ rt. $\measuredangle$s = 6 rt. $\measuredangle$s; if four angles were rt. $\measuredangle$s, the fifth angle would be a straight angle and the figure would be a rectangle. For the hexagon, six angles cannot be right angles, because the sum of the angles would be 6 rt. $\measuredangle$s in place of 8 rt. $\measuredangle$s; if five, or four angles were rt. $\measuredangle$s the polygon would be concave. Similarly for the octagon.

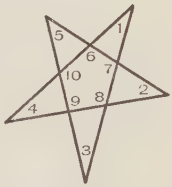
47. $(n - 2)\ 180° = 2 \cdot 360°$ or $n = 6$ (§§ 164, 166); the other answers are 8 and $2\ (x + 1)$.

Note: The last answer is a general solution which permits the construction of a formula.

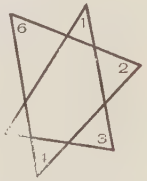
48. Respective angles are equal by § 134, $\angle 6 + \angle 7 + \angle 8 + \angle 9 + \angle 10 = 360°$ by the definition of a perigon in § 36. $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 360°$ (Sub.). (This proof assumes that $\measuredangle$ 6, 7, 8, 9, and 10 fill the angular space in the plane about the point. The proof is not rigorous.)

49. 0, 2, 5, 9, 20, $\frac{n}{2}(n - 3)$.

50. (*a*) For the irregular pentagram, $\angle 4 + \angle 2 + \angle 6 = 180°$, $\angle 5 + \angle 3 + \angle 7 = 180°$, $\angle 1 + \angle 4 + \angle 8 = 180°$, $\angle 2 + \angle 5 + \angle 9 = 180°$, $\angle 1 + \angle 3 + \angle 10 = 180°$ (§ 106); $\angle 6 + \angle 7 + \angle 8 + \angle 9 + \angle 10 = 540°$ (§ 164); $2\ \angle 1 + 2\angle 2 + 2\ \angle 3 + 2\ \angle 4 + 2\ \angle 5 = 360°$ (*Prins.* 5 and 6); $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 180°$ (*Prin.* 8).



(*b*) For the irregular hexagram, $\angle 1 + \angle 3 + \angle 5 = 180°$, $\angle 2 + \angle 4 + \angle 6 = 180°$ (§ 106); $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 360°$ (*Prin.* 5).



51. Solution A: $\frac{(n - 2)}{n}\ 180° = 135°$ (Given, § 167) $\frac{n - 2}{n} \cdot 4 = 3$ (Div. by 45°); $n = 8$.

Solution B: At any vertex, Int. $\angle$ + Ext. $\angle$ = 180° (*Prin.* 29); Int. $\angle$ = 135° (Given); Ext. $\angle$ = 45° (*Prin.* 6); All ext. $\measuredangle$ = 360° (§ 166); 360 ÷ 45 = 8 sides. The remaining answers are 18, 360, 6.

52. $\frac{360°}{9} = 40°$ (§ 167); the remaining answers are 36°, 9°, 3° 36′.

53. $\frac{(n - 2)\ 180}{n} = \frac{(3 - 2)\ 180°}{3} = 60°$ (§ 167); the remaining answers are 90°, 108°, 120°, 135°, 144°.

54. The sum of the angles of a hexagon is $(6 - 2)\ 180°$ (§ 164); $(n - 2)\ 180° = 3\ (6 - 2)\ 180°$ (Given); $n - 2 = 12$ or $n = 14$.

55. $(n - 2)\ 180° = 2 \cdot 180°$ (§§ 164, 166, Given); $n - 2 = 2$ or $n = 4$.

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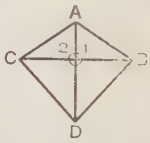
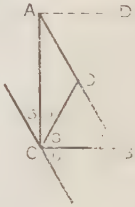
Miscellaneous Exercises

1. $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$ (§ 104); $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (Given); $\angle 1 + \angle 3 = 90°$ (Sub., *Prin.* 8); $90° = 180° - \angle 5$ (Sub., § 107) or $\angle 5 = 90°$.

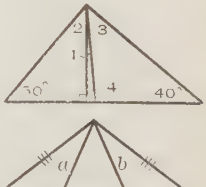


2. $\triangle ADC \cong \triangle ADB$ (*a.s.a.* = *a.s.a.*); $AC = AB$, $DC = DB$ (*Prin.* 38); $AD \perp CB$ (§ 171).

3. $\triangle ABC \cong \triangle DCB$ (*s.s.s.* = *s.s.s.*); $\angle 1 = \angle 2$ (*Prin.* 38); $OB = OC$ (§ 120); $OA = OD$ (*Prin.* 6); $\angle 4 = \angle 3$ (§ 80); $2\ \angle 3 = 2\ \angle 2$ (§ 107, Sub.); $\angle 3 = \angle 2$ (*Prin.* 8); $AD \parallel BC$ (§ 114).



4. Produce CO making $DO = CO$ (*Prin.* 11); draw AD and DB (*Prin.* 13); $\triangle AOD \cong \triangle BOC$ (*s.a.s.* = *s.a.s.*). $ADBC$ is a rect. (§ 140); $OA = OB = OC$ (§ 149, Ex. 4, p. 91); $\angle 1 = \angle 2$, $\angle 3 = \angle 4$ (§ 80); $\angle 1 = \angle 5$, $\angle 4 = \angle 6$ (§ 101); $\angle 2 = \angle 5$, $\angle 3 = \angle 6$ (Sub.).



5. $\angle 1 + \angle 2 = \angle 3$ (Given); $\angle 2 = 40°$ (§ 108); $\angle 3 = 40° + \angle 1$ (Sub.); $\angle 4 = (40° + \angle 1) + 40°$ (§ 110); $\angle 4 + \angle 1 = 90°$ (§ 108); $90° - \angle 1 = 80° + \angle 1$ (Sub.) or $\angle 1 = 5°$.

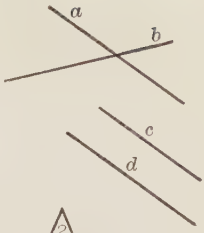
6. See Ex. 10, p. 84.

7. $a = b$ (*Prin.* 38).

8. Two lines parallel respectively to two intersecting lines intersect.

Note: In a proof by the Reductio ad Absurdum Method (§ 115), the assumption of the negative of the conclusion always produces an improper figure.

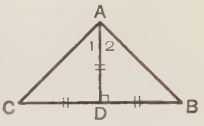
Given: a intersecting b, $a \parallel c$, and $b \parallel d$.
To prove c intersects d.
Proof: Suppose $c \parallel d$, then $d \parallel a$ (*Prin.* 16); $d \parallel b$ (Given); then $a \parallel b$ (*Prin.* 16), which is impossible. $\therefore$ c intersects d.



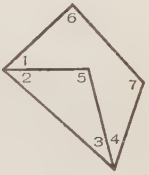
9. A restatement of § 168.

10. The words to be filled in are "one-half the angle." $\angle 2 = 90° - \angle 3$, $\angle 1 = 90° - \angle 4$ (§ 108); $\angle 4 = \angle 1 + \angle 3$ (§ 80); $\angle 1 = 90° - \angle 1 - \angle 3$ (Sub.) or $2\angle 1 = 90° - \angle 3$; $2\angle 1 = \angle 2$ (Sub.) or $\angle 1 = \frac{1}{2}\angle 2$.

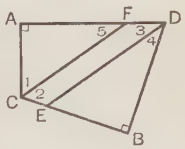
11. It is an isosceles right triangle. rt. $\triangle ADC \cong$ rt. $\triangle ADB$ (*l.l.* = *l.l.*); $AC = AB$ (*Prin.* 38); $\angle C + \angle 1 = 90°$ (§ 108); $\angle C = \angle 1$ (§ 80); $2\angle 1 = 90$ (Sub.); $\angle 1 = 45°$ (*Prin.* 8). In like manner, $\angle 2 = 45°$. $\angle 1 + \angle 2 = 90°$ (*Prin.* 5).



12. $2\angle 2 + 2\angle 3 + \angle 6 + \angle 7 = 360°$ (§ 164); $\angle 2 + \angle 3 + \frac{1}{2}(\angle 6 + \angle 7) = 180°$ (*Prin.* 8); $\angle 2 + \angle 3 + \angle 5 = 180°$ (§ 106); $\angle 2 + \angle 3 + \angle 5 = \angle 2 + \angle 3 + \frac{1}{2}(\angle 6 + \angle 7)$ (Sub.); $\angle 5 = \frac{1}{2}(\angle 6 + \angle 7)$ (*Prin.* 6).

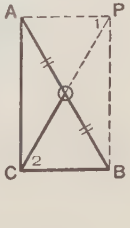


13. $\angle 1 + \angle 5 = 90°$ (§ 108); $\angle A + 2\angle 2 + \angle B + 2\angle 3 = 360°$ (§ 164); $\angle A + \angle B = 180°$ (Given and *Prin.* 5); $2\angle 2 + 2\angle 3 = 180°$ or $\angle 2 + \angle 3 = 90°$ (*Prins.* 6 and 8); $\angle 1 = \angle 2$ (Given); $\angle 1 + \angle 3 = 90°$ (Sub.); $\angle 5 = \angle 3$ (Sub.); $CF \parallel DE$ (§ 116).



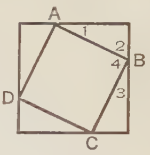
14. See Ex. 12. $\angle 6 = \angle 7 = 90°$ (Given); $\angle 5 = \frac{1}{2}(180°) = 90°$ (Sub.).

15. Make $OP = CO$ (*Prin.* 11); $\triangle AOP \cong \triangle BOC$ (*s.a.s.* = *s.a.s.*); $AP = CB$, $\angle 1 = \angle 2$ (*Prin.* 38); $AP \parallel CB$ (§ 114); $APBC$ is a rect. (§§ 152, *a*, 140); $CP = AB$ (Ex. 4, p. 91); $CO = \frac{1}{2}AB$ (*Prin.* 8).



16. Use Ex. 15 and $OA = OB = OC$ (Sub.).

17. Four triangles are congruent by *s.a.s.* = *s.a.s.* $AB = BC = CD = DA$ (*Prin.* 38); $ABCD$ is a $\square$ (§ 151); $\angle 1 = \angle 3$ (*Prin.* 38); $\angle 1 + \angle 2 = 90°$ (§ 108); $\angle 2 + \angle 3 = 90°$ (Sub.); $\angle 2 + \angle 3 + \angle 4 = 180°$ (§ 54); $\angle 4 = 90°$ (*Prin.* 6); $ABCD$ is a square (§§ 140, 141).



18. Use § 155 and add.

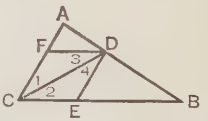
19. $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 360°$ (§ 36); $\angle 2 + \angle 2 + \angle 3 + \angle 4 + \angle 4 + \angle 3 = 360°$ (Sub.); $\angle 2 + \angle 3 + \angle 4 = \angle 2 + (\angle 3 + \angle 4) = 180°$ (*Prin.* 8); a is a straight line (*Prin.* 30).

20. Proof is like that of Ex. 1, p. 110 (Concurrent Lines).

21. $(n - 2)\ 180° = 6\ (2 \cdot 180°)$ (§§ 164, 166, Given); $(n - 2) = 12$ or $n = 14$.

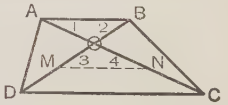
22. $AE = BF$ (*Prin.* 8); $AED \parallel BFC$ (§ 138); $ABFE$ is a $\square$ (§ 152, *a*); $AB \parallel EF \parallel DC$ (§ 138 and *Prin.* 16); $EF = AB = DC$ (§ 147 and Sub.) or $EF = \frac{1}{2}\ (AB + DC)$ (*Prins.* 5, 8).

23. $FD \parallel CE$, $DE \parallel FC$ (Given); $\angle 3 = \angle 2$, $\angle 4 = \angle 1$ (§ 101); $\angle 1 = \angle 2$ (Given); $\angle 1 = \angle 3$, $\angle 2 = 4$ (Sub.); $\triangle FDC$ and $\triangle EDC$ are congruent isos. $\triangle$ (*a.s.a.* = *a.s.a.*, § 120); $FD = DE = EC = CF$ (*Prin.* 38, Sub.); $FDEC$ is a rhombus (§§ 151, 139).



24. rt. $\triangle AFB \cong$ rt. $\triangle AFC$ (*hy.l.* = *hy.l.*); $CF = FB$ (*Prin.* 38); $CG = GF$, $BE = EF$ (§ 154); $CG = GF = FE = EB$ (*Prin.* 8).

25. Let M and N be the mid-points of OC and OD (*Prin.* 12); $MN \parallel DC$ and $MN = \frac{1}{2}\ DC$ (§ 155); $AB \parallel CD$ and $AB = \frac{1}{2}\ DC$ (Given); $AB = MN$ (Sub.); $\angle 1 = \angle 4$, $\angle 2 = \angle 3$ (§ 101); $\triangle AOB \cong \triangle NOM$ (*a.s.a.* = *a.s.a.*); $AO = ON = NC$ and $BO = OM = MD$ (*Prin.* 38 and Sub.).



CHAPTER TWO

CIRCLES—THEIR LINES AND ANGLES

Page 121, § 182

Note: The best modern usage considers a circle as a line and the circumference as the length of that line. This is consistent with the concept that the graph of $x^2 + y^2 = r^2$ is a closed curved line and not a portion of a plane.

Page 122, § 190

Note: The simple verb forms "has" and "have" are used instead of "intercepts" and "subtends" to express the relationship between a central angle and its arc and also between a chord and its arc.

Page 123, § 192

Suggestion: It is advisable to accustom the pupil to express both parts of a ratio in terms of a common unit. This will aid the pupil later in his course in making both terms of a ratio refer to corresponding parts of figures. The use of equal ratios in Chapters Two and Three offers a gradual approach to the proportions of Chapter Four.

PAGE 125

Note: The use of Principle 31 avoids an incommensurable form of proof. The teacher will observe an absence of incommensurable cases in the text.

PAGE 129

Suggestion: Radii are drawn as auxiliary construction lines in many proofs of Chapter Two. Such radii give an equality of parts in proving triangles congruent. Do not permit pupils to say merely, "Radii are equal." **Many mistakes are due to incomplete statements.**

The converse proof may be done as a sight exercise in class. Notice that the statements of many converse proofs are the same

as the statements used in the direct proofs, but occur in a different order and with different Facts.

If pupils cannot answer a "Why" of the Facts, do not give them the answer directly. Refer them to one of the Summaries. (See Index.)

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Exercises. Central Angles, Arcs, and Chords

1. The vertical angles are equal (*Prin.* 27); the opposite arcs are equal (§ 194).

2. $\angle 1 = \angle 2$ (Given); $\angle 3 = \angle 3$ (Iden.); $\angle 1 + \angle 3 = \angle 2 + \angle 3$ (Ax. of Add.); $\overset{\frown}{AD} = \overset{\frown}{CB}$ (§ 194).

3. $\overset{\frown}{AD} = \overset{\frown}{CB}$ (Given); $\overset{\frown}{CD} = \overset{\frown}{CD}$ (Iden.); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (Ax. of Subtraction); $\angle 1 = \angle 2$ (§ 194).

4. The radii are equal (*Prin.* 33); their halves are equal (*Prin.* 8); the connecting lines are equal (Given); the triangles are congruent (*s.s.s.* = *s.s.s.*); the central angles are equal (*Prin.* 38); the intercepted arcs are equal (§ 194).

5. Draw radii OA and OB (*Prin.* 13); $OA = OB$ (*Prin.* 33); $AC = CB$ (Given); $OC = OC$ (Iden.); $\triangle OCA \cong \triangle OCB$ (*s.s.s.* = *s.s.s.*); $\angle AOC = \angle BOC$ (*Prin.* 38); $\overset{\frown}{AD} = \overset{\frown}{DB}$ (§ 194).

6. $\overset{\frown}{PR} = \overset{\frown}{PS} = \overset{\frown}{RS}$ (Given); $PR = PS = RS$ (§ 196); $\angle P = \angle R = \angle S$ (§ 81).

7. $AC \perp BD$ (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DA}$ (§ 194); $AB = BC = CD = DA$ (§ 196); $ABCD$ is a $\square$ (§ 151); $OA = OB$ (*Prin.* 33); $\angle OAB = \angle OBA$ (§ 80); $\angle OAB + \angle OBA = 90°$ (§ 108); $2\ \angle OAB = 90°$ (Sub.); $\angle OAB = 45°$, and in like manner $\angle OAD = 45°$ (*Prin.* 8); $\angle DAB = 90°$ (*Prin.* 5); $ABCD$ is a square (§ 141).

8. $PRST$ is a square (Given); $PR = RS = ST = TP$ (§§ 141, 147, and Sub.); $\overset{\frown}{PR} = \overset{\frown}{RS} = \overset{\frown}{ST} = \overset{\frown}{TP} = \frac{1}{4}$ the circle (§ 196).

9. $\overset{\frown}{AB} = \overset{\frown}{CD}$ (Given); $\overset{\frown}{BC} = \overset{\frown}{BC}$ (Iden.); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (*Prin.* 5); $AC = BD$ (§ 196).

10. $AC = BD$ (Given); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (§ 196); $\overset{\frown}{BC} = \overset{\frown}{BC}$ (Iden.); $\overset{\frown}{AB} = \overset{\frown}{CD}$ (*Prin.* 6).

PAGE 134

Exercises. Bisected Chords and Arcs

1. Draw AD and DB (*Prin.* 13); $\overset{\frown}{AD} = \overset{\frown}{DB}$ (Given); $AD = DB$ (§ 196); $AC = CB$ (Given); $CD \perp AB$ (§ 171).

2. (Fig. of Ex. 4.) Draw radii OA and OC (*Prin.* 13); $OA = OC$ (*Prin.* 33); $\overset{\frown}{AB} = \overset{\frown}{BC}$ (Given); $AB = BC$ (§ 196); $OB \perp AC$ (§ 171).

3. Let CD be the chord; $AC = AD$ and $BC = BD$ (Given); AB is the $\perp$ bisector of CD (§ 171); AB is a diameter (§ 202).

4. Draw radii OA and OC (*Prin.* 13); $OA = OC$ (*Prin.* 33); $AB = BC$ (Given); OB bisects AC (§ 171).

5. OB is a radius (Given) and bisects AC (Given); $OB \perp AC$ (§ 201); $\overset{\frown}{AB} = \overset{\frown}{BC}$ (§ 200); $AB = BC$ (§ 196).

6. The perpendicular bisectors of the sides of an inscribed quadrilateral pass through the center of the circle (§ 202) and are concurrent at that point (§ 174).

7. Let OC intersect AB at P; $\triangle AOP \cong \triangle BOP$ (*s.a.s.* = *s.a.s.*); $AP = PB$ (*Prin.* 38); OC is the $\perp$ bisector of AB (§ 171); OC passes through the center (§ 202).

Locate the center by placing the instrument in different positions as indicated and drawing two or more diameters.

8. Draw AC, AD, BC, and BD (*Prin.* 13); $\overset{\frown}{AC} = \overset{\frown}{AD}$ and $\overset{\frown}{BC} = \overset{\frown}{BD}$ (Given); $AC = AD$ and $BC = BD$ (§ 196); AB is the $\perp$ bisector of CD (§ 171).

9. Draw AC intersecting the bisector at D (*Prin.* 13); $\triangle ABD \cong \triangle CBD$ (*s.a.s.* = *s.a.s.*); $AD = DC$ (*Prin.* 38); the bisector of $\angle B$ is also the perpendicular bisector of AC (§ 171) and is a diameter (§ 202).

10. Diameter GH bisects CD (Given); $GH \perp CD$ (§ 201); $CD \parallel AB$ (Given); $GH \perp AB$ (§ 99); GH bisects AB (§ 200).

11. $OB = OP$ (*Prin.* 33); $OR \parallel AP$ (Given); OR bisects AB (§ 154); $OR \perp AB$ (§ 201); $\overset{\frown}{BR} = \overset{\frown}{RA}$ (§ 200).

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Exercises. Chords and Their Distances from Centers

1. The chords are equally distant from the center (Given); the chords are equal (§ 203); their arcs are equal (§ 196).

2. The arcs are equal (Given); their chords are equal (§ 196); the chords are equally distant from the center (§ 203).

3. The sides of the inscribed square are equal (§§ 141, 147 and Sub.); and equally distant from the center (§ 203).

4. Draw $\perp$s to the chords from center O (*Prin.* 15); the rt. $\triangle$ formed are congruent (*hy.a.* = *hy.a.*); the $\perp$s from the center are equal (*Prin.* 38); $AB = BC$ (§ 203).

5. The inscribed triangle is equilateral (Given); the perpendiculars from the center to the sides are equal (§ 203).

6. The proof corresponds with the proof of Ex. 4.

7. Draw $\perp$s from the center O to AB and CD (*Prin.* 15); $AB = CD$ (Given); the $\perp$s from the center are equal (§ 203); the rt. $\triangle$ are congruent (*hy.l.* = *hy.l.*); $\angle 1 = \angle 2$ (*Prin.* 38).

8. The proof corresponds with the proof of Ex. 4.

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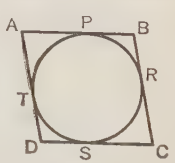
Exercises. Tangents

1. AR, RS, and CS are tangents (Given); $AR = BR$ and $CS = BS$ (§ 211); $AR + CS = RB + BS = RS$ (*Prin.* 5).

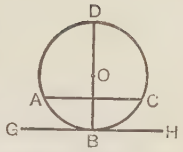
2. $ABCD$ is a circumscribed parallelogram (Given); its sides are tangent to the circle (§ 212); $AP = AT$, $BP = BR$, $DS = DT$, and $CS = CR$ (§ 211); $AB + DC = AD + BC$ (*Prin.* 5); $AB = DC$ and $AD = BC$ (§ 147); $2\,AB = 2\,AD$ (Sub.); $AB = AD$ (*Prin.* 8); $AB = AD = BC = DC$ (Sub.).



3. The proof corresponds with the first four steps of the proof of Ex. 2.

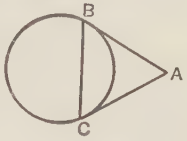
4. Draw radii OB and OC (*Prin.* 13); $OB = OC$ (*Prin.* 33); $AB = AC$ (§ 211); AO is the $\perp$ bisector of BC (§ 171).

5. Diameter $BD \perp GH$ (§ 208); $GH \parallel AC$ (Given); $BD \perp AC$ (§ 99); diameter BD bisects AC (§ 200).

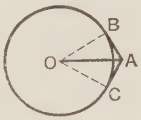


6. Draw radii of the inner circle to the points of tangency (*Prin.* 13); the tangents are perpendicular to the radii (§ 208); the radii are equal (*Prin.* 33); the tangents, which are chords of the outer circle, are equal (§ 203).

7. $AB = AC$ (§ 211); $\angle B = \angle C$ (§ 80); $\angle B + \angle C = 120°$ (§ 107); $2\ \angle B = 120°$ (Sub.); $\angle B = \angle C = \angle A$ (*Prin.* 8 and Sub.); $BC = AB = AC$ (§ 121).

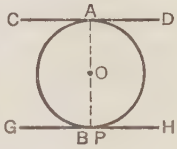


8. $AB = AC$; $\angle BAO = \angle CAO = 60°$ (§ 211 and *Prin.* 8); draw OB and OC (*Prin.* 13); $\angle B = \angle C = 90°$ (§ 208); $\angle BOA = \angle COA = 30°$ (§ 108); $OA = 2\ AB$ (§ 158); $OA = AB + AB = AB + AC$ (Sub.).



9. The tangents are perpendicular to the diameter (§ 208) and parallel (§ 98).

10. $CD \parallel GH$ (Given); draw OA (*Prin.* 13); $OA \perp CD$ (§ 208); produce AO to meet GH at P (*Prin.* 11); $OP \perp GH$ (§ 99); draw OB (*Prin.* 13); $OB \perp GH$ (§ 208); OP and OB coincide (*Prin.* 15) and AOP is a diameter.



11. See Ex. 5.

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Exercises. Common Tangents

1. HINT: The circles do not intersect and one lies without the other.

2. HINT: The circles are tangent externally.

3. HINT: The circles intersect.

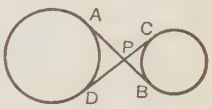
4. HINT: The circles are tangent internally.

5. HINT: The circles are not tangent and one lies within the other.

6. $OA = OC$ and $OB = OC$ (§ 211); $OA = OC = OB$ (Sub.).

7. The proof corresponds with the proof of Ex. 6.

8. Produce AB and CD to meet at O (*Prin.* 11); $OA = OC$ and $OB = OD$ (§ 211); $AB = CD$ (*Prin.* 6).



9. $PA = PD$ and $PB = PC$ (§ 211); $AB = CD$ (*Prin.* 5).

10. $PR = PA = PB = \frac{1}{2}\ AB$ and $RS = CS = SD = \frac{1}{2}\ CD$ (§ 211); $PR + RS = \frac{1}{2}\ AB + \frac{1}{2}\ CD$ (*Prin.* 5); $AB = CD$ (Ex. 8); $PS = \frac{1}{2}\ AB + \frac{1}{2}\ AB = AB = CD$ (Sub.).

11. Let the common internal tangent through C intersect the common external tangent AB at P (§ 216); $PC = PB = PA$ (§ 211); $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (§ 80); $\angle 1 + \angle 2 + \angle 3 + \angle 4 =$ st. $\angle$ (§ 106); $2\ \angle 2 + 2\ \angle 3 =$ st. $\angle$ (Sub.); $\angle 2 + \angle 3 = \angle ACB =$ rt. $\angle$ (*Prin.* 8).

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Exercises. Angles Formed by Tangents and Chords

1. $\angle BCD = \angle ACE$ (Given); $\overset{\frown}{CD} = \overset{\frown}{CE}$ (§ 220); $CD = CE$ (§ 196).

2. $CD = CE$ (Given); $\overset{\frown}{CD} = \overset{\frown}{CE}$ (§ 196); $\angle BCD = \angle ACE$ (§ 220).

3. $\angle ACD =$ numerically $\frac{1}{2}\ \overset{\frown}{AC}$ and $\angle BCD =$ numerically $\frac{1}{2}\ \overset{\frown}{BC}$ (§ 219); $\angle BCD = \frac{1}{2}\ \angle ACD$ (Given); $\frac{1}{2}\ \overset{\frown}{BC} = \frac{1}{2}$ of $\frac{1}{2}\ \overset{\frown}{AC}$ (Sub.); $\overset{\frown}{BC} = \frac{1}{2}\ \overset{\frown}{AC}$ (*Prin.* 7).

4. $OA = OB$ (§ 211); $\angle A = \angle B$ (§ 80).

5. $PS = PR$ (Given); $\overset{\frown}{PS} = \overset{\frown}{PR}$ (§ 196); $\angle 1 = \angle 2$ (§ 220).

6. Let the common internal tangent pass through O (§ 216); $\angle 1 = \angle 2$ (*Prin.* 27); $\angle 1 =$ numerically $\frac{1}{2}\ \overset{\frown}{OA}$ and $\angle 2 =$ numerically $\frac{1}{2}\ \overset{\frown}{OB}$ (§ 219); $\frac{1}{2}\ \overset{\frown}{OA} =$ numerically $\frac{1}{2}\ \overset{\frown}{OB}$ (Sub.); $\overset{\frown}{OA} =$ numerically $\overset{\frown}{OB}$ (*Prin.* 7).

7. $\overset{\frown}{AP} =$ numerically $\overset{\frown}{PB}$ (Ex. 6); $\angle O =$ numerically $\overset{\frown}{AP}$ and $\angle C =$ numerically $\overset{\frown}{PB}$ (*Prin.* 31); $\angle O = \angle C$ (Sub.).

8. $\angle BPC$ = numerically $\frac{1}{2}\overset{\frown}{PC}$ and $\angle BPD$ = numerically $\frac{1}{2}\overset{\frown}{PD}$ (§ 219); $\angle BPC = \angle BPD$ (Iden.); $\frac{1}{2}\overset{\frown}{PC}$ = numerically $\frac{1}{2}\overset{\frown}{PD}$ (Sub.); $\overset{\frown}{PC}$ = numerically $\overset{\frown}{PD}$ (*Prin.* 7).

9. $\overset{\frown}{PA}$ = numerically $\overset{\frown}{PA'}$ (Ex. 8); $\angle O$ = numerically $\overset{\frown}{PA}$ and $\angle O'$ = numerically $\overset{\frown}{PA'}$ (*Prin.* 31); $\angle O = \angle O'$ (Sub.); $OA \parallel O'A'$ (§ 116).

10. Draw OC as in Fig. for Ex. 7 (*Prin.* 13); OC passes through P (§ 215, B); $\angle O = \angle C$ (Ex. 7); $OA \parallel BC$ (§ 114).

Page 151, §224

Note: The sequence of proofs of the text makes it possible to prove Proposition 10 without the usual three cases.

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Exercises. Inscribed Angles

1. $\angle 1 = \angle 2$ (Given); $\overset{\frown}{BC} = \overset{\frown}{CD}$ (§ 227).

2. Let $\widehat{x}$ be the arc of inscribed angle A and central angle B; let $\widehat{y}$ be the arc of inscribed angle C and central angle D; $\angle A - \angle C = 10°$ (Given); $\angle A$ = numerically $\frac{1}{2}\widehat{x}$ and $\angle C$ = numerically $\frac{1}{2}\widehat{y}$ (§ 224); $\angle A - \angle C = \frac{1}{2}\widehat{x} - \frac{1}{2}\widehat{y} = 10°$ (*Prin.* 6); $\widehat{x} - \widehat{y} = 20°$ (*Prin.* 7); $\angle B$ = numerically $\widehat{x}$ and $\angle D$ = numerically $\widehat{y}$ (*Prin.* 31); $\angle B - \angle D = 20°$ (Sub.).

3. $\overset{\frown}{PB} = \frac{1}{2}\overset{\frown}{AB}$ (Given); $\angle C$ = numerically $\frac{1}{2}\overset{\frown}{AB}$ and $\angle PAB$ = numerically $\frac{1}{2}\overset{\frown}{PB}$ (§ 224); $2\ \angle PAB$ = numerically $\overset{\frown}{PB}$ (*Prin.* 7); $2\ \angle PAB = \angle C$ (Sub.)

4. $AB \perp BC$ and $CD \perp BC$ (Given); draw AC and BD (*Prin.* 13); $\angle A = \angle D$ (§ 225); rt. $\triangle ABC \cong$ rt. $\triangle BCD$ (*a.l.* = *a.l.*); $AB = CD$ (*Prin.* 38).

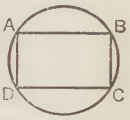
5. (*a*) $\angle P = \angle P$ (Iden.); $\angle B = \angle D$ (§ 227); $\angle BCP = \angle DAP$ (§ 111). (*b*) $\angle B = \angle D$ and $\angle A = \angle C$ (§ 227); $\angle AOB = \angle COD$ (*Prin.* 27).

6. $\angle A$ = numerically $\frac{1}{2}\overset{\frown}{BCD}$ and $\angle C$ = numerically $\frac{1}{2}\overset{\frown}{DAB}$ (§ 224); $\angle A + \angle C$ = numerically $\frac{1}{2}(\overset{\frown}{BCD} + \overset{\frown}{DAB})$ (*Prin.* 5); $\overset{\frown}{BCD} + \overset{\frown}{DAB} = 360°$ (§ 191); $\angle A + \angle C = \frac{1}{2}$ of $360° = 180°$ (Sub.).

7. The angle opposite the 60° angle equals 180° − 60° = 120° and the angle opposite the 70° angle equals 180° − 70° = 110° (Ex. 6).

8. Draw AC and BD (*Prin.* 13); $OA = OB$ (Given); $\angle C = \angle D$ and $\angle A = \angle B$ (§ 227); $\triangle AOC \cong \triangle BOD$ (*s.a.a.* = *s.a.a.*); $OC = OD$ (§ 80).

9. $ABCD$ is a $\square$ (Given); $AB = CD$ and $AD = BC$ (§ 147); $\overset{\frown}{AB} = \overset{\frown}{CD}$ and $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 196); $\overset{\frown}{AB} + \overset{\frown}{AD} = \overset{\frown}{CD} + \overset{\frown}{BC}$ (*Prin.* 5); $\overset{\frown}{AB} + \overset{\frown}{AD} + \overset{\frown}{CD} + \overset{\frown}{BC} = 360°$ (§ 191); $2(\overset{\frown}{CD} + \overset{\frown}{BC}) = 360°$ (Sub.); $\overset{\frown}{CD} + \overset{\frown}{BC} = \overset{\frown}{BCD} = 180°$ (*Prin.* 8); $\angle A$ = numerically $\frac{1}{2}\,\overset{\frown}{BCD}$ (§ 224); $\angle A = \frac{1}{2}$ of 180° = 90° (Sub.); $ABCD$ is a rectangle (§ 140).

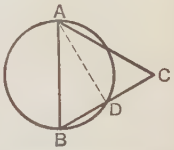


10. $ABCD$ is a square (Given); $AD = DC = BC$ (§§ 141, 147 and Sub.); $\overset{\frown}{AD} = \overset{\frown}{DC} = \overset{\frown}{BC}$ (§ 196); $\angle APD = \angle DPC = \angle BPC$ (§ 227).

11. $\overset{\frown}{AC} = \overset{\frown}{CB}$ (Given); CD is a diameter (Given); $\overset{\frown}{DAC} = \overset{\frown}{DBC}$ (*Prin.* 36); $\overset{\frown}{DA} = \overset{\frown}{DB}$ (*Prin.* 6); $DA = DB$ (§ 196).

12. The method is based upon the converse of § 226.

13. Draw AD (*Prin.* 13); $\overset{\frown}{ADB}$ is a semicircle (Given); $\angle ADB$ = rt. $\angle$ (§ 226); rt. $\triangle ADB \cong$ rt. $\triangle ADC$ (*hy.l.* = *hy.l.*); $BD = DC$ (*Prin.* 38).



14. $\angle 1$ = numerically $\frac{1}{2}(\hat{b} + \hat{c} + \hat{d} + \hat{e})$, $\angle 3$ = numerically $\frac{1}{2}(\hat{d} + \hat{e} + \hat{f} + \hat{a})$ and $\angle 5$ = numerically $\frac{1}{2}(\hat{f} + \hat{a} + \hat{b} + \hat{c})$; $\angle 1 + \angle 3 + \angle 5$ = numerically $\frac{1}{2}(2\hat{a} + 2\hat{b} + 2\hat{c} + 2\hat{d} + 2\hat{e} + 2\hat{f}) = \hat{a} + \hat{b} + \hat{c} + \hat{d} + \hat{e} + \hat{f} = 360°$.

15. $\angle A = 40°$, $\angle B = 60°$, and $\angle C = 80°$; AX, the bisector of $\angle A$, makes $\overset{\frown}{BX} = \overset{\frown}{CX} = 40°$; in like manner, $\overset{\frown}{CY} = \overset{\frown}{AY} = 60°$ and $\overset{\frown}{BZ} = \overset{\frown}{AZ} = 80°$; $\angle X$ = numerically $\frac{1}{2}(\overset{\frown}{AZ} + \overset{\frown}{AY}) = \frac{1}{2}(80° + 60°) = \frac{1}{2}(140°) = 70°$; $\angle Y$ = numerically $\frac{1}{2}(\overset{\frown}{BX} + \overset{\frown}{BZ}) = \frac{1}{2}(40° + 80°) = \frac{1}{2}(120°) = 60°$; $\angle Z$ = numerically $\frac{1}{2}(\overset{\frown}{CX} + \overset{\frown}{CY}) = \frac{1}{2}(40° + 60°) = \frac{1}{2}(100°) = 50°$.

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Exercises. Angles Formed by Intersecting Chords

1. $\angle 1$ = numerically $\frac{1}{2}(\overset{\frown}{x} + \overset{\frown}{y})$; (*a*) $\angle 1 = 90°$; (*b*) $\angle 1 = 88°$; (*c*) $y = 80°$; (*d*) $x = 52°$.

2. $\angle AOD$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC})$; $90°$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + 80°)$; $180° = \overset{\frown}{AD} + 80°$; $\overset{\frown}{AD} = 100°$; $\overset{\frown}{DC} = 95°$; $\angle A = 87\frac{1}{2}°$; $\angle B = 97\frac{1}{2}$; $\angle C = 92\frac{1}{2}°$; $\angle D = 82\frac{1}{2}°$.

3. Use the figure of Ex. 2; $AC \perp BD$ (Given); $\angle AOD$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC})$ (§ 229); $90° = \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC})$ (Sub.); $180° = \overset{\frown}{AD} + \overset{\frown}{BC}$ (*Prin.* 7).

4. If the sum of the opposite arcs formed by intersecting chords equals a semicircle, the chords are perpendicular. Use the figure of Ex. 2. $\overset{\frown}{AD} + \overset{\frown}{BC} = 180°$ (Given); $\angle AOD$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC})$ (§ 229); $\angle AOD = \frac{1}{2}(180°) = 90°$ (Sub.); $AC \perp BD$ (§ 27).

5. $BC = 110°$; $\angle A = 95°$; $\angle B = 90°$; $\angle C = 85°$; $\angle D = 90°$; $\angle 1 = 75°$; $\angle 2 = 105°$; AC is a diameter (*Prin.* 36).

6. $\angle CDB + \angle BCA + \angle 2 = 55° + 35° + 105° = 195°$.

7. Use the figure of Ex. 5. $3x + 4x + 5x + 6x = 18x = 360°$; $x = 20°$; $3x = \overset{\frown}{AB} = 60°$; $4x = \overset{\frown}{BC} = 80°$; $5x = \overset{\frown}{CD} = 100°$; $6x = \overset{\frown}{DA} = 120°$; $\angle A = 90°$; $\angle B = 110°$; $\angle C = 90°$; $\angle D = 70°$; $\angle 1 = 80°$; $\angle 2 = 100°$.

8. $\angle 1$ = numerically $\frac{1}{2}(\overset{\frown}{CD} + \overset{\frown}{AH})$ and $\angle 2$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BH})$ (§ 229); $\overset{\frown}{CD} = \overset{\frown}{AD}$ and $\overset{\frown}{AH} = \overset{\frown}{BH}$ (Given); $\overset{\frown}{CD} + \overset{\frown}{AH} = \overset{\frown}{AD} + \overset{\frown}{BH}$ (*Prin.* 5); $\angle 1 = \angle 2$ (Sub. and *Prin.* 8); $AE = AG$ (§ 120).

9. $\angle AOC$ = numerically $\overset{\frown}{AC}$ and $\angle BOD$ = numerically $\overset{\frown}{BD}$ (*Prin.* 31); $\angle AOC + \angle BOD$ = numerically $(\overset{\frown}{AC} + \overset{\frown}{BD})$ (*Prin.* 5); $\angle APC$ = numerically $\frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$ (§ 229); $2\ \angle APC$ = numerically $(\overset{\frown}{AC} + \overset{\frown}{BD})$ (*Prin.* 7); $\angle AOC + \angle BOD = 2\ \angle APC$ (Sub.).

10. Let OA and CD intersect at R and let OD and AB intersect at S; $\angle PRO = \angle PSO = 90°$ (Given); $\angle CPB + \angle PSO + \angle DOA + \angle PRO = 360°$ (§ 164); $\angle CPB + \angle DOA = 180°$ (*Prin.* 6);

$\angle CPB = 2\ \angle DOA$ (Given); $3\ \angle DOA = 180°$ (Sub.); $\angle DOA = 60°$ (*Prin.* 8); $\overset{\frown}{AD} = 60°$ (*Prin.* 31).

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Exercises. Angles Formed by Secants and Tangents

1. Let the intercepted arcs be $\overset{\frown}{x}$ and $\overset{\frown}{y}$. $\angle A$ = numerically $\frac{1}{2}(\overset{\frown}{x} - \overset{\frown}{y})$; $2\ \angle A$ = numerically $(\overset{\frown}{x} - \overset{\frown}{y})$; $\overset{\frown}{x} + \overset{\frown}{y} = 360°$; (*a*) $\overset{\frown}{x} - \overset{\frown}{y} = 80°$; $\overset{\frown}{x} + \overset{\frown}{y} = 360°$; $\overset{\frown}{x} = 220°$; $\overset{\frown}{y} = 140°$; (*b*) $\overset{\frown}{x} = 240°$; $\overset{\frown}{y} = 120°$; (*c*) $\overset{\frown}{x} = 280°$; $\overset{\frown}{y} = 80°$.

2. $\overset{\frown}{x} - \overset{\frown}{y} = 180°$; $\overset{\frown}{x} + \overset{\frown}{y} = 360°$; $\overset{\frown}{x} = 270°$; $\overset{\frown}{y} = 90°$.

3. $\angle O$ = numerically $\frac{1}{2}(\overset{\frown}{CD} - \overset{\frown}{AB})$; $2\ \angle O$ = numerically $(\overset{\frown}{CD} - \overset{\frown}{AB})$. (*a*) $\angle O = 20°$; (*b*) $\angle O = 25°$; (*c*) $\overset{\frown}{CD} = 230°$; (*d*) $\overset{\frown}{AB} = 8°$.

4. Let the intercepted arcs be $\overset{\frown}{x}$ and $360° - \overset{\frown}{x}$. $\angle A$ = numerically $\frac{1}{2}(360° - \overset{\frown}{x} - \overset{\frown}{x}) = \frac{1}{2}(360° - 2\overset{\frown}{x}) = 180° - \overset{\frown}{x}$. If $\overset{\frown}{x} = 50°$, $\angle A = 130°$. If $\overset{\frown}{x} = 60°$, $\angle A = 120°$. If $\overset{\frown}{x} = 40°$, $\angle A = 140°$. If $\overset{\frown}{x} = x°$, $\angle A = 180° - x°$.

5. $\angle AOD$ = numerically $\overset{\frown}{AD}$ and $\angle BOC$ = numerically $\overset{\frown}{BC}$ (*Prin.* 31); $\angle AOD - \angle BOC$ = numerically $(\overset{\frown}{AD} - \overset{\frown}{BC})$ (*Prin.* 6); $\angle P$ = numerically $\frac{1}{2}(\overset{\frown}{AD} - \overset{\frown}{BC})$ (§ 230); $2\ \angle P$ = numerically $(\overset{\frown}{AD} - \overset{\frown}{BC})$ (*Prin.* 7); $\angle AOD - \angle BOC = 2\ \angle P$ (Sub.).

6. $\angle 1$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BC})$ (§ 229); $\angle 2$ = numerically $\frac{1}{2}(\overset{\frown}{AD} - \overset{\frown}{BC})$ (§ 230); $\angle 1 + \angle 2$ = numerically $\overset{\frown}{AD}$ (*Prin.* 5); $\overset{\frown}{AD} = 60°$ (Given); $\angle 1 + \angle 2 = 60°$ (Sub.).

7. $2x + 3x = 360°$; $x = 72°$; $2x = 144°$; $3x = 216°$; $\angle A = \frac{1}{2}(216° - 144°) = 36°$.

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Note: This section need not be given special emphasis. It is not a part of the suggested Minimum Course. (See pp. vii, viii.)

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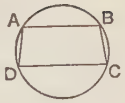
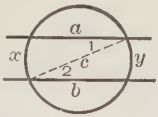
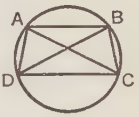
Exercises. Parallel Lines and Their Arcs



1. $AB \parallel CD$ (§ 137); $AD = BC$ (§ 232); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 196); $ABCD$ is an isosceles trapezoid (§ 137).



2. Draw c (*Prin.* 13); $\widehat{x} = \widehat{y}$ (Given); $\angle 1 = \angle 2$ (§ 227); $a \parallel b$ (§ 114).



3. $AC = BD$ (Given); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (§ 196); $\overset{\frown}{AB} = \overset{\frown}{AB}$ (Iden.); $\overset{\frown}{BC} = \overset{\frown}{AD}$ (*Prin.* 6); $BC = AD$ (§ 196); $AB \parallel CD$ (Ex. 2); $ABCD$ is an isosceles trapezoid (§ 137).

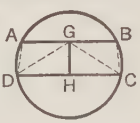
4. $ABCD$ is an isosceles trapezoid. $AD = BC$ (Given); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 196); $AB \parallel CD$ (Ex. 2); $ABCD$ is an isosceles trapezoid (§ 137).

5. $AB \parallel CD$; $\overset{\frown}{AD} = \overset{\frown}{BC} = \widehat{x}$: $2\widehat{x} + 104° + 156° = 360°$; $2\widehat{x} + 260° = 360°$; $2\widehat{x} = 100°$; $\widehat{x} = \overset{\frown}{AD} = \overset{\frown}{BC} = 50°$.

6. $\angle x$ = numerically $\frac{1}{2}(\overset{\frown}{CD} - \overset{\frown}{AB})$; $\angle x = \frac{1}{2}(156° - 104°) = \frac{1}{2}$ of $52° = 26°$.

7. $ABC \parallel HGD$ (Given); $\angle ABG = \angle BGD$ (§ 101); $\angle H + \angle ABG = 180°$ and $\angle C + \angle BGD = 180°$ (Ex. 6, p. 152); $\angle H = \angle C$ and in like manner $\angle A = \angle D$ (*Prin.* 28); $ACDH$ is a $\square$ (§ 152, C); $AC = HD$ (§ 147).

8. $AB \parallel CD$ (Given); draw AD, BC, GD, and GC (*Prin.* 13); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 232); $AD = BC$ (§ 196); $AG = GB$ (Given); $\overset{\frown}{DC} = \overset{\frown}{DC}$ (Iden.); $\overset{\frown}{ADC} = \overset{\frown}{BCD}$ (*Prin.* 5); $\angle A = \angle B$ (§ 227); $\triangle ADG \cong \triangle BCG$ (*s.a.s.* = *s.a.s.*); $GD = GC$ (*Prin.* 38); $DH = HC$ (Given); $GH \perp DC$ (§ 171); $GH \perp AB$ (§ 99).



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Note: If it seems desirable, Inequalities §§ 384, 385, Constructions §§ 389–413, and Loci §§ 431–442 may be taken at the end of Chapter Two. However, teachers will find a gain in continuity of development and in unity of subject matter by considering each

of these topics as a whole at a later period. Read the last half of p. viii of the Preface.

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Supplementary Exercises. Central Angles, Arcs, and Chords

1. $AD = BC$ (Given); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 196); $\overset{\frown}{AB} = \overset{\frown}{AB}$ (Iden.); $\overset{\frown}{BD} = \overset{\frown}{AC}$ (*Prin.* 5); $BD = AC$ (§ 196).

2. In the annexed figure, if $BD = AC$ show that $AD = BC$. $BD = AC$ (Given); $\overset{\frown}{BD} = \overset{\frown}{AC}$ (§ 196); $\overset{\frown}{AB} = \overset{\frown}{AB}$ (Iden.); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (*Prin.* 6); $AD = BC$ (§ 196).

3. $AB = BC = CD$ (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ (§ 196); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (*Prin.* 5); $\angle AOC = \angle BOD$ (§ 194).

4. $\overset{\frown}{AD} = \overset{\frown}{CD}$ (Given); BD is a diameter (Given); $\overset{\frown}{BAD} = \overset{\frown}{BCD}$ (*Prin.* 36); $\overset{\frown}{AB} = \overset{\frown}{BC}$ (*Prin.* 6); $AB = BC$ (§ 196).

5. $AB = CD$ (Given); $\overset{\frown}{AB} = \overset{\frown}{CD}$ (§ 196); $\overset{\frown}{BC} = \overset{\frown}{BC}$ (Iden.); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (*Prin.* 5); $AC = BD$ (§ 196).

6. $\triangle ABC \cong \triangle BDC$ (Ex. 5 and *s.s.s.* = *s.s.s.*).

7. $AB = BC = CD$ = etc. (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ = etc. (§ 196); $\overset{\frown}{AC} = \overset{\frown}{BD}$ = etc. (*Prin.* 5); $AC = BD$ (§ 196).

8. Draw OP (*Prin.* 13); rt. $\triangle PRO \cong$ rt. $\triangle PSO$ (*hy.l.* = *hy.l.*); $\angle POR = \angle POS$ (*Prin.* 38); $\overset{\frown}{AP} = \overset{\frown}{PS}$ (§ 194).

9. Draw OC (*Prin.* 13); $\triangle GOC \cong \triangle HOC$ (*s.s.s.* = *s.s.s.*); $\angle GOC = \angle HOC$ (*Prin.* 38); $\overset{\frown}{AC} = \overset{\frown}{BC}$ (§ 194).

10. $AB = BC = CD$ = etc. (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ = etc. (§ 196); $\angle 1 = \angle 2 = \angle 3$ = etc. (§ 194); $\angle 1 = 60°$ (*Prin.* 8); $\angle 7 + \angle 8 = 120°$ (§ 107); $OA = OB$ = etc. (*Prin.* 33); $\angle 7 = \angle 8 = 60°$ (§ 80 and *Prin.* 8); $\angle 1 = \angle 7 = \angle 8 = 60°$ (Sub.); $OA = OB = AB$ and likewise for the other triangles (§ 121).

11. $AB = CD$ (Given); $\overset{\frown}{AB} = \overset{\frown}{CD}$ (§ 196); $AB \parallel CD$ (Given); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (§ 232).

12. Since the chord is identical for two equal circles, the arcs are equal by § 196.

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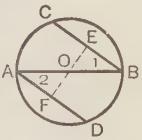
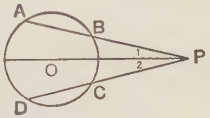
Supplementary Exercises. Chords and Their Distances from Centers

13. $BC \parallel AD$ (Given); draw FE through center $O \perp CB$ (*Prin.* 15); $FE \perp AD$ (§ 99); $\angle 1 = \angle 2$ (§ 101); $OA = OB$ (*Prin.* 33); rt. $\triangle OEB \cong$ rt. $\triangle OFA$ (*hy. a.* = *hy. a.*); $OE = OF$ (*Prin.* 38); $CB = AD$ (§ 203).

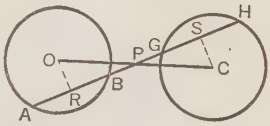


14. See Ex. 6, p. 137.

15. Draw OP and GR both $\perp ABCD$ (*Prin.* 15); $OP \parallel GR$ (§ 98); $PR \parallel OG$ (Given); $OP = GR$ (§ 148); $AB = CD$ (§ 203).

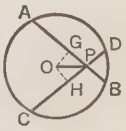


16. Draw OR and CS both $\perp AH$ (*Prin.* 15); $OP = PC$ (Given); rt. $\triangle ORP \cong$ rt. $\triangle CSP$ (*hy. a.* = *hy. a.*); $OR = CS$ (*Prin.* 38); $AB = GH$ (§ 203).



17. Draw $OP \perp AD$ (*Prin.* 15); $AP = PD$ and $BP = PC$ (§ 200); $AB = CD$ (*Prin.* 6).

18. Draw $OG \perp AB$ and $OH \perp CD$ (*Prin.* 15); $AB = CD$ (Given); $OG = OH$ (§ 203); rt. $\triangle OGP \cong$ rt. $\triangle OHP$ (*hy. l.* = *hy. l.*); $\angle OPA = \angle OPC$ (*Prin.* 38).



19. Draw $OG \perp AB$ and $OH \perp CD$ (*Prin.* 15); draw OP (*Prin.* 13); $AB = CD$ (Given); $OG = OH$ (§ 203); rt. $\triangle OGP \cong$ rt. $\angle OHP$ (*hy. l.* = *hy. l.*); $GP = HP$ (*Prin.* 38); $AG = \frac{1}{2} AB$ and $DH = \frac{1}{2} CD$ (§ 200); $AG = DH$ (*Prin.* 8); $PA = PD$ (*Prin.* 5).

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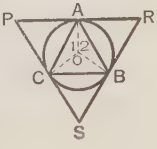
Supplementary Exercises. Tangents

20. $AC = AB$ (Given); $\overset{\frown}{AC} = \overset{\frown}{AB}$ (§ 196); draw radii OA, OB, and OC (*Prin.* 13); $\angle 1 = \angle 2$ (§ 194); $\measuredangle OAR$, OAP, OBR, and OCP are rt. $\measuredangle$ (§ 208); $\angle 1 + \angle P = 180°$ and $\angle 2 + \angle R = 180°$ (§ 164 and *Prin.* 6); $\angle P = \angle R$ (*Prin.* 28); $SP = SR$ (§ 120).



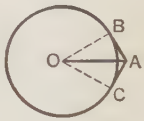
21. $DH = DB$ and $CH = CA$ (§ 211); $PD = PD$ and $PC = PC$ (Iden.); $PD + DC + PC = PB + PA$ (*Prin.* 5).

22. $OA \perp PA$ and $OB \perp PB$ (§ 208); $\angle P + \angle A + \angle O + \angle B = 360°$ (§ 164); $\angle A = \angle B = 90°$ (§ 208); $\angle P + \angle O + 180° = 360°$ (Sub.); $\angle P + \angle O = 180°$ (*Prin.* 6).

23. Draw radii OA, OB, and OH (*Prin.* 13); rt. $\triangle OAC \cong$ rt. $\triangle OHC$ (*hy. l. = hy. l.*); $\angle AOC = \angle COH = \frac{1}{2} \angle AOH$ and likewise $\angle DOH = \frac{1}{2} \angle BOH$ (§ 77); $\angle COD = \frac{1}{2} \angle O$ (*Prin.* 5); $\frac{1}{2} \angle P = \frac{1}{2} \angle P$ (Iden.); $\frac{1}{2} \angle P + \angle COD = \frac{1}{2} (\angle P + \angle AOB)$ (*Prin.* 5); $\angle P + \angle AOB = 180°$ (Ex. 22); $\frac{1}{2} \angle P + \angle COD = \frac{1}{2} (180°) = 90°$ (Sub.).

24. Draw radii r and r' (*Prin.* 13); $a = b$ (§ 211); $abrr'$ is a square (§ 141); $a = b = r = r'$ (§ 147); $a + b = r + r' =$ diameter D (*Prin.* 5); $e = f$ and $d = c$ (§ 211); $e + d = f + c$ (*Prin.* 5); $f + a + b + c = f + c + D$ (Sub.); $f + a + b + c - (e + d) = D$ (*Prin.* 6).

25. Draw radii OC and OB (*Prin.* 13); $OC \perp AC$ and $OB \perp BA$ (§ 208); $AB = AC$ (§ 211); $OA = AB + AC$ (Given); $OA = AB + AB = 2\,AB$ (Sub.); $\angle BAO = 60°$ (§ 159) and in like manner, $\angle CAO = 60°$; $\angle BAC = 60° + 60° = 120°$ (*Prin.* 5).



26. $PA = PB$ (§ 211); $OA = OB$ (*Prin.* 33); $\angle 1 = \angle 3$ (§ 80); $PA \perp OA$ and $PB \perp OB$ or $\angle OAP = 90°$ and $\angle OBP = 90°$ (§ 208); $\angle OAP + \angle OBP = \angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$ (*Prin.* 5); $\angle 2 + \angle 4 + \angle P = 180°$ (§ 106); $\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle 2 + \angle 4 + \angle P$ (Sub.); $\angle 1 + \angle 3 = \angle P$ (*Prin.* 6); $2\,\angle 1 = \angle P$ (Sub.).

27. $OA = OB$ (*Prin.* 33); $PA = PB$ (§ 211); $OP \perp AB$ (§ 171); OP bisects arc AB (§ 200).

28. $OA = OB$ (*Prin.* 33); $PA = PB$ (§ 211); the $\perp$ bisector of AB passes through O and P (§ 170).

29. $\angle 1 = \angle 2 = \frac{1}{2} \angle P$ and $\angle 3 = \angle 4 = \frac{1}{2} \angle Q$ (§ 211); $AP \parallel CQ$ (Given); $\angle P + \angle Q = 180°$ (§ 104); $\frac{1}{2} \angle P + \frac{1}{2} \angle Q = 90°$ (*Prin.* 8); $\angle 2 + \angle 3 = 90°$ (Sub.); $\angle O + \angle 2 + \angle 3 = 180°$ (§ 106); $\angle O = 90°$ (*Prin.* 6).

30. $OA \perp OB$ (Given); $\angle 1 = \frac{1}{2}\ \angle A$ and $\angle 2 = \frac{1}{2}\ \angle B$ (§ 211); $\angle 1 + \angle 2 = 90°$ (§ 108); $\frac{1}{2}\ \angle A + \frac{1}{2}\ \angle B = 90°$ (Sub.); $\angle A + \angle B = 180°$ (*Prin.* 7); $AP \parallel BR$ (§ 117).

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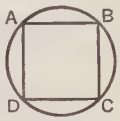
Supplementary Exercises. Inscribed Angles

31. $\angle 1 = \angle 4$, $\angle 2 = \angle 7$, $\angle 3 = \angle 6$, and $\angle 5 = \angle 8$ (§ 227).

32. $\triangle OAD$ and $\triangle OBC$ are mutually equiangular, likewise $\triangle OAB$ and $\triangle OCD$ (Ex. 31).

33. Arc $EABCD = 296°$; arc $DE = 64°$; $\angle A$ = numerically $\frac{1}{2}$ arc $BCDE = \frac{1}{2}$ of $226° = 113°$; $\angle B = 104°$; $\angle C = 99°$; $\angle D = 106°$; $\angle E = 118°$. Check: $113° + 104° + 99° + 106° + 118° = 540°$ (§ 164).

34. $AB = BC = CD = AD$ (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{AD}$ (§ 196); $4\ \overset{\frown}{AB} = 360°$ (§ 191 and Sub.); $\overset{\frown}{AB} = 90°$ (*Prin.* 8); $\angle A$ = numerically $\frac{1}{2}$ arc BCD (§ 224); $\angle A = \frac{1}{2}$ of $180° = 90°$ (Sub.); $ABCD$ is a square (§ 141).



35. $2x + 4x + 6x = 360°$; $x = 30°$; $2x = 60°$; $4x = 120°$; $6x = 180°$; $\angle A = 30°$; $\angle B = 60°$; $\angle C = 90°$; the triangle is a 30°-60°-90° triangle.

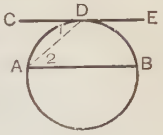
36. $5x + 7x + 12x = 360°$; $24x = 360°$; $x = 15°$; $5x = 75°$; $7x = 105°$; $12x° = 180°$; $\angle A = 37\frac{1}{2}°$; $\angle B = 52\frac{1}{2}°$; $\angle C = 90°$.

37. $\angle R$ = numerically $\frac{1}{2}\ (\overset{\frown}{AB} + \overset{\frown}{BC})$, $\angle S$ = numerically $\frac{1}{2}\ (\overset{\frown}{BC} + \overset{\frown}{AC})$, $\angle T$ = numerically $\frac{1}{2}\ (\overset{\frown}{AC} + \overset{\frown}{AB})$ (§ 224); $\angle R + \angle S + \angle T$ = numerically $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{AC} = 360°$ (*Prin.* 5).

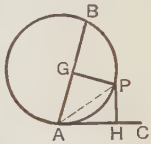
38. The angles are respectively equal (Given); their arcs are equal (§ 227); the sides of the triangles are respectively equal (§ 196); the triangles are congruent. (*s.s.s.* = *s.s.s.*).

39. Draw AD and BC (*Prin.* 13); $AC = BD$ (Given); $\overset{\frown}{AC} = \overset{\frown}{BD}$ (§ 196); $\overset{\frown}{AB} = \overset{\frown}{AB}$ (Iden.); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (*Prin.* 6); $AD = BC$ (§ 196); $\angle D = \angle C$ and $\angle A = \angle B$ (§ 227); $\triangle OAD \cong \triangle OBC$ (*a.s.a.* = *a.s.a.*); $OA = OB$ and $OD = OC$ (*Prin.* 38).

40. (*a*) $\overset{\frown}{AD} = \overset{\frown}{DB}$ (Given); $\angle 1 = \angle 2$ (§ 228); $CE \parallel AB$ (§ 114). (*b*) Draw a perpendicular to the tangent at the point of tangency (*Prin.* 15); the perpendicular passes through the center (§ 209); the perpendicular to the tangent is also perpendicular to the chord (Ex. 2, p. 134); the tangent and the chord are parallel (§ 98).



41. Produce OP to meet the circle at D (*Prin.* 11); $OP \perp AB$ (Given); $\overset{\frown}{AD} = \frac{1}{2}\overset{\frown}{AB}$ (§ 200); $\angle O$ = numerically $\overset{\frown}{AD}$ (*Prin.* 31); $\angle C$ = numerically $\frac{1}{2}\overset{\frown}{AB}$ (§ 224); $\angle O = \angle C$ (Sub.).



42. $\overset{\frown}{AP} = \overset{\frown}{BP}$ (Given); $\angle BAP = \angle PAC$ (§ 228); $PG = PH$ (§ 172).

43. $\angle 1 = \angle 2$ (Given); $AB \parallel DE$ (Given); $\angle 1 = \angle 3$ (§ 101); $\angle 2 = \angle 3$ (Sub.); $\overset{\frown}{BE} = \overset{\frown}{DC}$ (§ 227); $\overset{\frown}{CE} = \overset{\frown}{CE}$ (Iden.); $\overset{\frown}{BC} = \overset{\frown}{DE}$ (*Prin.* 5); $BC = DE$ (§ 196).

44. The circles are equal (Given); $AB = AB$ (Iden.); the right-hand arc AB = the left-hand arc AB (§ 196); $\angle G = \angle H$ (§ 227); $BG = BH$ (§ 120).

45. Draw OG and AO (*Prin.* 13); $AO \perp$ the common external tangent (§ 208); AO is a diameter (§ 209); $OG \perp AB$ (§ 226); $AG = GB$ (§ 200).

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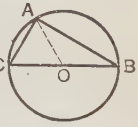
Miscellaneous Exercises

1. See Ex. 3 of Bisected Chords and Arcs, p. 134.

2. Draw OA (*Prin.* 13); $OA = OB = OC$ (Ex. 16, p. 120); the circle through B and C with center O passes also through A (*Prin.* 34).



3. This exercise is a double application of Ex. 2.

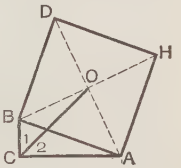
4. Draw BT (*Prin.* 13); $\angle A = 90°$ (Given); $\angle A$ = numerically $\frac{1}{2}\,BDT$ (§ 224); $\frac{1}{2}\,BDT = 90°$ (Sub.); $BDT = 180°$ and BOT is a diameter (*Prin.* 7); $OT = OB$ (*Prin.* 33); $\angle ROT = \angle BOS$ (*Prin.* 27); AT and $BS \perp AB$ (Given); $AT \parallel BS$ (§ 98); $\angle RTO = \angle OBS$ (§ 101); $\triangle RTO \cong \triangle OBS$ (*a.s.a.* = *a.s.a.*); $RO = OS$ (*Prin.* 38); $OC = OD$ (*Prin.* 33); $CR = SD$ (*Prin.* 6).

5. Draw radii OP and $O'T$ (*Prin.* 13); OP and $O'T \perp$ tangent PT (§ 208); $OP \parallel O'T$ (§ 98); $\angle POB = \angle TO'D$ (§ 102); $\angle POB =$ numerically $\overset{\frown}{PB}$ and $\angle TO'D =$ numerically $\overset{\frown}{TD}$ (*Prin.* 31); $\overset{\frown}{PB} =$ numerically $\overset{\frown}{TD}$ (Sub.); $\angle PAB =$ numerically $\frac{1}{2}\overset{\frown}{PB}$ and $\angle TCD =$ numerically $\frac{1}{2}\overset{\frown}{TD}$ (§ 224); $\frac{1}{2}\overset{\frown}{PB} =$ numerically $\frac{1}{2}\overset{\frown}{TD}$ (*Prin.* 8); $\angle PAB = \angle TCD$ (Sub.); $AP \parallel CT$ and in like manner $PB \parallel TD$ (§ 116).

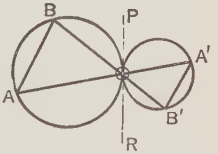
6. $BS \perp AS$ and $AR \perp RB$ (Given); draw a circle whose diameter is AB (*Prin.* 32); R and S lie on the circle whose diameter is AB (Ex. 2); $\angle ABS = \angle ARS$ (§ 227).

7. Draw radius OD (*Prin.* 13); $OD = OC$ (*Prin.* 33); $OH = \frac{1}{2}OC$ (Given); $OH = \frac{1}{2}OD$ (Sub.); $RD \perp OC$ (Given); $\angle D = 30°$ (§ 159); $AB \perp OC$ (Given); $RD \parallel AB$ (§ 98); $\angle D = \angle 1$ (§ 101); $\angle 1 = 30°$ (Sub.); $\angle COA = 90°$ (§ 27); $\angle COD = 60°$ (*Prin.* 6); $\angle COD = 2\ \angle 1$ (Sub.); $\angle COD =$ numerically $\overset{\frown}{CD}$ and $\overset{\frown}{AD} =$ numerically $\angle 1$ (*Prin.* 31); $\overset{\frown}{CD} = 2\ \overset{\frown}{AD}$ (Sub.).

8. Draw diagonals AD and BH of square $ABDH$ (*Prin.* 13); $\angle AOB = 90°$ (§ 171); $\angle BCA = 90°$ (Given); a circle whose diameter is AB passes through C and O (Ex. 3); $AD = BH$ (§ 77); $OB = OA$ (§ 149); $\overset{\frown}{OB} = \overset{\frown}{OA}$ (§ 196); $\angle 1 = \angle 2$ (§ 227).



9. Let POR be the common internal tangent (§ 214); $\angle B = \angle AOR$ and $\angle B' = \angle A'OP$ (§ 228); $\angle AOR = \angle A'OP$ (*Prin.* 27); $\angle B = \angle B'$ (Sub.); $AB \parallel A'B'$ (§ 114).



10. $\angle AOE = \angle A + \angle C$ (§ 110); draw radius OB (*Prin.* 13); $OA = OB$ (*Prin.* 33); $\angle A = \angle ABO$ (§ 79); $\angle ABO = \angle BOC + \angle C$ (§ 110); $BC = OD$ (Given); $OD = OB$ (*Prin.* 33); $BC = OB$ (Sub.); $\angle BOC = \angle C$ (§ 79); $\angle AOE = \angle A + \angle C = \angle ABO + \angle C = \angle BOC + \angle C + \angle C = \angle C + \angle C + \angle C = 3\ \angle C$ (Sub.).

11. See Theorem B, § 215.

12. Draw DE, TE, and TD (*Prin.* 13); $\overset{\frown}{CD} = \overset{\frown}{DT}$ (Given); $\angle CTD = \angle DTS$ (§ 228); let DE intersect CT at P; $DP = DS$ and likewise $EP = ER$ (§ 172); $DS + ER = DP + EP = DE$

(*Prin.* 5); DE is a diameter (p. 134, Ex. 3 of Bisected Chords and Arcs); $DS + ER$ = a diameter (Sub.).

13. $\angle 1 = \angle 2$ = rt. $\angle$ (Given); $\angle 1$ = numerically $\frac{1}{2}(\overset{\frown}{RC} + \overset{\frown}{AB})$ and $\angle 2$ = numerically $\frac{1}{2}(\overset{\frown}{CS} + \overset{\frown}{AB})$ (§ 229); $\frac{1}{2}(\overset{\frown}{RC} + \overset{\frown}{AB}) = \frac{1}{2}(\overset{\frown}{CS} + \overset{\frown}{AB})$ (Sub.); $\overset{\frown}{RC} + \overset{\frown}{AB} = \overset{\frown}{CS} + \overset{\frown}{AB}$ (*Prin.* 7); $\overset{\frown}{RC} = \overset{\frown}{CS}$ (*Prin.* 6).

14. $AB \parallel CD$ (Given); draw radii OA and OB (*Prin.* 13); $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$ (§ 101); $OA = OB$ (*Prin.* 33); $\angle 1 = \angle 2$ (§ 80); $\angle 3 = \angle 4$ (Sub.); $\angle 3$ = numerically $\overset{\frown}{AD}$ and $\angle 4$ = numerically $\overset{\frown}{BC}$ (*Prin.* 31); $\overset{\frown}{AD} = \overset{\frown}{BC}$ (Sub.).

15. $\angle MGH$ = numerically $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{MB})$ (§ 229); $\angle C$ = numerically $\frac{1}{2}\overset{\frown}{DM} = \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{AM})$ (§ 224); $\overset{\frown}{AM} = \overset{\frown}{MB}$ (Given); $(\overset{\frown}{AD} + \overset{\frown}{AM}) = (\overset{\frown}{AD} + \overset{\frown}{MB})$ (*Prin.* 5); $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{AM}) = \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{MB})$ (*Prin.* 8); $\angle MGH = \angle C$ and likewise $\angle MHG = \angle D$ (Sub.); $\angle M = \angle M$ (Iden.).

16. $\angle D$ = numerically $\frac{1}{2}(\overset{\frown}{MB} + \overset{\frown}{BC})$ (§ 224); $\angle AHC$ = numerically $\frac{1}{2}(\overset{\frown}{ADC} + \overset{\frown}{MB})$ (§ 229); $\angle D + \angle AHC$ = numerically $\frac{1}{2}(\overset{\frown}{MB} + \overset{\frown}{BC} + \overset{\frown}{ADC} + \overset{\frown}{MB})$ (*Prin.* 5); $\overset{\frown}{MB} = \overset{\frown}{AM}$ (Given); $\angle D + \angle AHC = \frac{1}{2}(360°) = 180°$ (Sub.).

17. $\angle BPR = \frac{1}{2}$ of $64° = 32°$; $\angle RPT = \frac{1}{2}(126° + 78°) = 102°$; $\overset{\frown}{PT} = 92°$; $\angle APT = \frac{1}{2}$ of $92° = 46°$. Check: $32° + 102° + 46° = 180°$.

18. $\angle A = \frac{1}{2}(268° - 92°) = 88°$; $\angle B = \frac{1}{2}(296° - 64°) = 116°$; $\angle C = \frac{1}{2}(234° - 126°) = 54°$; $\angle D = \frac{1}{2}(282° - 78°) = 102°$. Check: $88° + 116° + 54° + 102° = 360°$.

CHAPTER THREE

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Note: It will be observed by the teacher that the text uses no "incommensurable cases" of proof. This has been brought about by the introduction of two Principles (*Prin.* 31 and *Prin.* 39) and by the placement of Areas before Proportions. Such procedures are not novel nor experimental. The first of these is sanctioned in leading syllabi and the second has been used by certain American and French authors with noteworthy success in the past.

The introduction of Areas before Proportions makes it possible to give a direct proof not only for Proposition 12, p. 190 (in place of one which is usually nothing more than a numerical illustration and not a proof), but also of Proposition 13, p. 193 (in place of an indirect proof). In addition, the use of ratios begun in Chapter Two and continued with Areas in Chapter Three gives a far better preparation for the work of Proportional Magnitudes than is found in many texts. Too frequently pupils are exposed to the difficulties of the usual Book III without adequate preparation or an appreciation of the meaning of equal ratios.

PAGE 171

Exercises. Areas of Rectangles

1. (*a*) $n = (16 \times 12 \times 24 \times 12) \div 6 \times 6 = 1536$. (*b*) $n = 864$

2. $x + y = 13$; $xy = 36$; $x = 9$, $y = 4$.

3. $(a + b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

4. (*a*) Let x = the width; $2x$ = the length; $6x$ = the perimeter; $6x = 12$; $x = 2$; $2x = 4$; $2x^2$ = the area (S) = 8. (*b*) $S = 18$; (*c*) $S = 22\frac{2}{9}$; (*d*) $S = \frac{p^2}{18}$.

PAGE 172, § 244

Suggestion: Special emphasis should be given Corollary 2 as it is a Tool in this text and is used in a number of later proofs. Corollary 9 finds frequent use in many numerical calculations.

PAGE 173

Exercises. Areas of Triangles

1. $S = \dfrac{9 \times 6}{2} = 27$; $S = 35$; $S = 96$.

2. $\dfrac{\triangle T}{\triangle T'} = \dfrac{24 \times 16}{36 \times 20} = \dfrac{8}{15}$.

3. (*a*) $h = 12$; (*b*) $h = 9$; (*c*) $h = 6$; (*d*) $h = 8$.

4. $S = \frac{1}{2}\, bh$ and $S = \frac{1}{2}$ the hypotenuse $\times$ the altitude upon the hypotenuse (§ 242); bh = the hypotenuse $\times$ the altitude upon the hypotenuse (Sub. and *Prin.* 7).

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Exercises. Areas of Parallelograms

1. Let R = the area of the rectangle and P = the area of the parallelogram; $R = P$; $R = bh$; $P = bh'$; $bh = bh'$; $h = h'$.

2. Let the bases be b and $2\,b$ and the altitudes $2\,a$ and a, respectively; $P = b \times 2\,a$ and $P' = 2\,b \times a$; $P = 2\,ab = P'$.

3. $P = bh$; $T = \frac{1}{2}\, bh'$; $P = T$; $bh = \frac{1}{2}\, bh'$; $h = \frac{1}{2}\, h'$.

4. $ABCD = AH \times CD = CG \times AD$ (§ 253, Sub.).

5. $T = \frac{1}{2} \times 6 \times 4 = 12$; $P = 10 \times 6 = 60$; $\dfrac{T}{P} = \dfrac{12}{60} = \dfrac{1}{5}$.

6. Use the fig. of § 253; draw AC (*Prin.* 13); $\triangle ABC = \frac{1}{2}\, BC \times h$ and $\triangle ACD = \frac{1}{2}\, AD \times h$ (§ 242). $BC = AD$ (§ 147); $\triangle ABC = \frac{1}{2}\, BC \times h = \triangle ACD$ (Sub.).

7. $CH \parallel BA$, $CB \parallel DA$, and $GB \parallel HA$ (Given); $ABCD$ and $ABGH$ are ⧈ (§ 138); their altitudes on base AB are equal (§ 148); $ABCD = ABGH$ (§ 256).

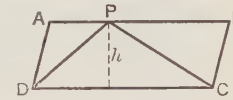
8. $CH \parallel BA$, $CB \parallel DA$, $GB \parallel HA$ (Given); $CD = BA = GH$ (§ 148, Sub.); $DG = DG$ (Iden.); $CG = DH$ (*Prin.* 5); $\triangle BCG = \triangle ADH$ (§ 246).

Optional proof: $ABCH = ABCH$ (Iden.); $ABGH = ABCD$ (Ex. 7); $\triangle BCG = \triangle ADH$ (*Prin.* 6).

9. $DE = \frac{1}{2}\, CD$ (Given); $\triangle ADE = \frac{1}{2} \cdot \frac{1}{2}\, CD \cdot h = \frac{1}{4}\, h \cdot CD$ (§ 242); $\square ABCD = h \cdot CD$ (§ 253); $\dfrac{\triangle ADE}{\square ABCD} = \dfrac{\frac{1}{4}\, h \cdot CD}{h \cdot CD} = \frac{1}{4}$ (*Prin.* 8).

10. Let the diagonals intersect at O; $AC \perp BD$ (§ 171); $\triangle ABD = \frac{1}{2} BD \times AO$ and $\triangle BCD = \frac{1}{2} BD \times OC$ (§ 242); $ABCD = \frac{1}{2} BD (AO + OC) = \frac{1}{2} BD \times AC$ (*Prin.* 5).

11. $AO = OC$ and $BO = OD$ (§ 149); $\triangle OAB = \triangle OAD = \triangle ODC = \triangle OBC$ (§ 245); $\triangle ABC = \triangle ADC = \triangle ABD = \triangle BCD = \frac{1}{2} \square ABCD$ (§ 146).



12. Draw altitude h (*Prin.* 15); $\triangle PCD = \frac{1}{2} h \times CD$ (§ 242); $\square ABCD = h \times CD$ (§ 253); $\triangle PCD = \frac{1}{2} \square ABCD$ (Sub.).

13. $S = \frac{1}{2} \times 6 \times 8 = 5h$; $5h = 24$; $h = 4.8$.

14. $S = \frac{1}{2} \times 6 \times 8 = \frac{1}{2} \times 10h$; $5h = 24$; $h = 4.8$.

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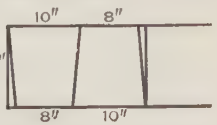
Exercises. Areas of Trapezoids

1. $S = \frac{1}{2} h (b + b') = \frac{1}{2} \times 3 (20 + 15) = 52.5$; $\triangle ADE = \frac{1}{2} \times 20 \times 12 = 120$; $\triangle ABC = \frac{1}{2} \times 15 \times 9 = 67.5$; $S = 52.5$.

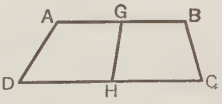
2. Use the fig. of § 257; $\triangle ACD = \frac{1}{2} bh$ and $\triangle ABC = \frac{1}{2} b'h$ (§ 242); $\dfrac{\triangle ACD}{\triangle ABC} = \dfrac{b}{b'}$ (*Prin.* 8).

3. Each side $= \frac{1}{2} \times 11(10 + 8) = 99$; the four sides of each basket $= 4 \times 99 = 396$; the base of each basket $= 8 \times 8 = 64$; the surface of each basket $= 396 + 64 = 460$; the total surface of 50 baskets $= 50 \times 460$ sq. in. $= 23{,}000$ sq. in. $= 159.7$ sq. ft.

4. Each pair of sides require 18″ board length with a shortage of 1″ at the end of the board; each board $= 12' = 144''$; $144 \div 18 = 8$ pairs, with a shortage of $1'' = 7\frac{1}{2}$ pairs $= 15$ sides; $50 \times 4 = 200$ sides; $200 \div 15 = 13\frac{1}{3}$ or 14 boards; $13 \times 15 = 195$ sides from 13 boards, leaving 5 sides or 46″ to be cut from the 14th board.



5. $ABCD$ is a trapezoid with $AG = GB$ and $DH = HC$ (Given); let h be the perpendicular between AB and DC: $AGHD = \frac{1}{2} h (AG + DH)$ and $GBCH = \frac{1}{2} h (GB + HC)$ (§ 257); $AGHD = GBCH$ (Sub.).



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Note: In the majority of the exercises of this text involving the use of Tool XIX, it will be found that the equal altitudes of a pair of triangles are a common altitude. In such cases the corresponding bases are segments of the same line.

Special attention should be given the exercises on this page as they are intended for practice with Tool XIX and for preparation for Propositions 6, 12, 13, 14, and 15. It will be found that Ex. 8 develops the elements of proof which are used in Proposition 6.

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Exercises. Ratios of Lines of Triangles

1. (*a*) $\frac{\triangle 1}{\triangle ABC} = \frac{CD}{CB}$; (*b*) $\frac{\triangle 2}{\triangle ABC} = \frac{DB}{CB}$; (*c*) $\frac{\triangle 1}{\triangle 2} = \frac{CD}{DB}$.

2. (*a*) $\frac{\triangle 1}{\triangle ABC} = \frac{CG}{CB}$; (*b*) $\frac{\triangle 2}{\triangle ABC} = \frac{GB}{CB}$; (*c*) $\frac{\triangle 1}{\triangle 2} = \frac{CG}{GB}$.

3. (*a*) $\frac{\triangle 1}{\triangle ABC} = \frac{AH}{AB}$; (*b*) $\frac{\triangle 2}{\triangle ABC} = \frac{HB}{AB}$; (*c*) $\frac{\triangle 1}{\triangle 2} = \frac{AH}{HB}$.

4. (*a*) $\frac{OS}{OC} = \frac{\triangle 3}{\triangle 4}$; (*b*) $\frac{OS}{OC} = \frac{\triangle 2}{\triangle 1}$; (*c*) $\frac{\triangle 2}{\triangle 1} = \frac{\triangle 3}{\triangle 4}$; (*d*) $\frac{\triangle CRS}{\triangle BRS} = 1$ (§ 246, *Prin.* 2); (*e*) $\frac{\triangle 2}{\triangle RSB} = \frac{RO}{RB}$.

5. (*a*) $\frac{\triangle 1}{\triangle BCG} = \frac{BH}{BC}$; (*b*) $\frac{\triangle BDH}{\triangle 1} = \frac{BD}{BG}$; (*c*) $\frac{BD}{BG} = \frac{\triangle BDH}{\triangle 1} = \frac{\triangle BCD}{\triangle BCG}$; (*d*) $\frac{BH}{CH} = \frac{\triangle 1}{\triangle GCH} = \frac{\triangle BDH}{\triangle CDH}$; (*e*) $\frac{\triangle GDO}{\triangle GOH} = \frac{OD}{OH}$; (*f*) $\frac{\triangle HOC}{\triangle GOH} = \frac{OC}{OG}$.

6. $\frac{\triangle BDH}{\triangle 1} = \frac{BD}{BG}$; $\frac{\triangle 1}{\triangle BCG} = \frac{BH}{BC}$; $\frac{\triangle BDH}{\triangle BCG} = \frac{BD \times BH}{BG \times BC}$ (*Prin.* 7).

7. $\frac{\triangle GDO}{\triangle GOH} = \frac{DO}{OH}$; $\frac{\triangle HOC}{\triangle GOH} = \frac{OC}{OG}$; $\frac{\triangle GDO}{\triangle HOC} = \frac{DO \times OG}{OC \times OH}$ (*Prin.* 8).

8. (*b*) Draw a line from $R \perp CS$; (*c*) $\frac{\triangle 1}{\triangle 2} = \frac{CO}{OS}$; (*d*) draw a line from $S \perp RB$; (*e*) $\frac{\triangle 2}{\triangle 3} = \frac{RO}{OB}$; (*f*) $\frac{\triangle 1}{\triangle 3} = \frac{OC \times OR}{OB \times OS}$ (*Prin.* 7).

PAGE 180, § 262

Suggestion: Emphasis should be given Tool XX and the exercises which illustrate it. Tool XX is the basis of some later propositions.

PAGE 180, § 263

Note: The general topic of Proportion is given a full treatment for the benefit of pupils who have not studied the subject previously in algebra. If pupils have a knowledge of proportion, §§ 263–276 may be given a rapid review.

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Exercises. Ratios of Products of Lines

1. (*a*) $\frac{\triangle ADC}{\triangle ADB} = \frac{AC \times AD}{AB \times AD} = \frac{AC}{AB}$; (*b*) $\frac{\triangle ACD}{\triangle ACB} = \frac{AC \times CD}{AC \times CB} = \frac{CD}{CB}$; (*c*) $\frac{\triangle ABD}{\triangle ABC} = \frac{AB \times BD}{AB \times BC} = \frac{BD}{BC}$.

2. (*a*) $\frac{\triangle BCD}{\triangle CDE} = \frac{8 \times 3}{3 \times 5} = \frac{8}{5}$; (*b*) $\frac{\triangle CDE}{\triangle ACE} = \frac{3 \times 5}{5 \times 9} = \frac{1}{3}$; (*c*) $\frac{\triangle BCD}{\triangle ACE} = \frac{8 \times 3}{5 \times 9} = \frac{8}{15}$.

3. (*a*) $\frac{\triangle ADC}{\triangle ADB} = \frac{CD \times DA}{BD \times DA} = \frac{CD}{BD} = \frac{\frac{1}{2}BD}{BD} = \frac{1}{2}$; (*b*) $\frac{\triangle ADC}{\triangle ADB} = \frac{\frac{1}{3}BD}{BD} = \frac{1}{3}$; (*c*) $\frac{\triangle ADC}{\triangle ADB} = \frac{\frac{3}{4}BD}{BD} = \frac{3}{4}$.

4. (*a*) $\frac{\triangle ABD}{\triangle ABC} = \frac{BD \times AB}{BC \times AB} = \frac{BD}{BC}$. If $CD = \frac{1}{2}BD$, $BC = \frac{1}{2}BD$; if $CD = \frac{1}{3}BD$, $BC = \frac{2}{3}BD$; if $CD = \frac{3}{4}BD$, $BC = \frac{1}{4}BD$. $\frac{\triangle ABD}{\triangle ABC} = \frac{BD}{\frac{1}{2}BD} = \frac{2}{1}$; (*b*) $\frac{\triangle ABD}{\triangle ABC} = \frac{BD}{\frac{2}{3}BD} = \frac{3}{2}$; (*c*) $\frac{\triangle ABD}{\triangle ABC} = \frac{BD}{\frac{1}{4}BD} = \frac{4}{1}$.

5. (*a*) $\frac{\triangle CDE}{\triangle BDA} = \frac{2 \times 3}{4 \times 6} = \frac{1}{4}$; (*b*) $\frac{\triangle CDE}{\triangle BDA} = \frac{3 \times 5}{6 \times 10} = \frac{1}{4}$; (*c*) $\frac{\triangle CDE}{\triangle BDA} = \frac{2 \times 5}{8 \times 8} = \frac{5}{32}$.

PAGE 189

Exercises

1. (*a*) $18 : 6 = 15 : 5$; $90 = 90$; (*b*) $7 : 3 = 14 : 6$; $42 = 42$; (*c*) $25 : 5 = 20 : 4$; $100 = 100$.

2. (*a*) $6 : 6 = 5 : 5$; $30 = 30$; (*b*) $1 : 3 = 2 : 6$; $6 = 6$; (*c*) $15 : 5 = 12 : 4$; $60 = 60$.

3. Yes; by subtraction.

4. Yes; by addition.

5. (*a*) $12 : 9 = 8 : 6$; $72 = 72$; (*b*) $8 : 12 = 6 : 9$; $72 = 72$; (*c*) $20 : 8 = 15 : 6$; $120 = 120$; (*d*) $4 : 8 = 3 : 6$; $24 = 24$.

6. (*a*) $10 : 6 = 5 : x$ (by addition); (*b*) $2 : 6 = 2 : x$ (by subtraction); (*c*) $10 : 4 = 10 : 5 - x$ (by addition); (*d*) $6 : 3 - x = 8 : 3$ (by addition).

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Exercises. Ratios of Line Segments

1. (*a*) $6 : 8 = 9 : 12$; (*b*) $9 : 12 = 6 : 8$; (*c*) $8 : 6 = 12 : 9$.

2. $\frac{\triangle T}{\triangle T'} = \frac{6}{8} = \frac{3}{4}$; if the common side is t, $\frac{\triangle T}{\triangle T'} = \frac{9t}{12t} = \frac{3}{4}$.

3. (*a*) $20 : 30 = 6 : 9$; (*b*) $6 : 9 = 20 : 30$; (*c*) $30 : 20 = 9 : 6$.

4. $3x + 4x = 15$; $7x = 15$; $x = 2\frac{1}{7}$; $3x = 6\frac{3}{7}$; $4x = 8\frac{4}{7}$.

5. $12 : 15 = 4 : 5$; $4x + 5x = 9$; $9x = 1$; $4x = 4$; $5x = 5$.

6. $9 : 15 = 3 : 5$; $3x + 5x = 12$; $8x = 12$; $x = 1\frac{1}{2}$; $3x = 4\frac{1}{2}$; $5x = 7\frac{1}{2}$.

7. (*a*) $\frac{12}{18} = \frac{2}{3}$; (*b*) $\frac{18}{20} = \frac{9}{10}$; (*c*) $\frac{20}{12} = \frac{5}{3}$.

8. The bisector to side 20 divides it in the ratio $\frac{2}{3}$; its smaller segment equals 8; the bisector is divided into segments whose ratio is $\frac{8}{12} = \frac{2}{3}$.

9. $8 : 12 = 2 : 3 = a : b = 10 : 15 = 12 : 18 = 14 : 21 =$ etc.

10. Let the triangles from left to right be A, B, and C and the segments of side 10 be a, b, and c; $\frac{\triangle A}{\triangle B} = \frac{6 \times 4}{4 \times 3} = \frac{2}{1}$; $\frac{\triangle B}{\triangle C} = \frac{4 \times 3}{3 \times 5} = \frac{4}{5}$; $\triangle C : \triangle B : \triangle A = 5 : 4 : 8$; $\triangle C : \triangle B : \triangle A = c : b : a = 5 : 4 : 8 = 5x : 4x : 8x$; $5x + 4x + 8x = 10$; $17x = 10$; $x = \frac{10}{17}$; $5x = 2\frac{16}{17}$; $4x = 2\frac{6}{17}$; $8x = 4\frac{12}{17}$.

11. Draw $m \parallel t$ (*Prin.* 14); produce b (*Prin.* 11); $d : e = n : b$ (§ 279); $d : e = a : b$ (Given); $n = a$ (§ 270); $\angle ma$ or $\angle 3 = \angle mn$ or $\angle 4$ (§ 79); $\angle 3 = \angle 1$ (§ 101); $\angle 4 = \angle 2$ (§ 102); $\angle 1 = \angle 2$ (Sub.).

12. $b = 15$; $n = 9$; $e = 10$; $d = 6$; $b = 30$; $e = 20$; $d = 12$; $a = 18$. $\angle 1 = \angle 2$ and $m \parallel t$ (Given); $a : b = d : e$ (§ 286); $n : b = d : e$ (§ 279); $a = n$ (§ 270).

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Supplementary Exercises. Areas of Rectangles

1. (*a*) $x\ (x + 4) = 96$; $x^2 + 4x = 96$; $x^2 + 4x - 96 = 0$; $(x + 12)(x - 8) = 0$; $x - 8 = 0$; $x = 8$; $x + 4 = 12$. (*b*) $x\ (x + 10) = 96$; $x = 6$; $x + 10 = 16$.

2. (*a*) $x + 2x + 3 = 12$; $x = 3$; $2x + 3 = 9$; $S = 27$; (*b*) $x = 4$; $2x + 3 = 11$; $S = 44$.

3. $x + 15 = 40$; $x = 25$; $S = 275$; if the yard were square, $4y = 80$; $y = 20$; $S' = 20 \times 20 = 400$; $S' - S = 400 - 375 = 25$.

4. Let x = the width of the frame; $2x + 10$ = the outer width; $2x + 15$ = the outer length; $(2x + 10)(2x + 15) = 4x^2 + 50x + 150$ = the entire area; $10 \times 15 = 150$ = the area of the picture; $4x^2 + 50x$ = the area of the frame; $4x^2 + 50x = 150$; $2x^2 + 25x - 75 = 0$; $(2x - 5)(x + 15) = 0$; $2x - 5 = 0$; $x = 2\frac{1}{2}$.

5. $S = rt + t\ (s - t) = rt + st - t^2 = t\ (r + s - t)$. (*a*) $S = \frac{1}{4}(8\frac{3}{4}) = 2\frac{3}{16}$; (*b*) $S = \frac{1}{2}(16\frac{1}{2}) = 8\frac{1}{4}$; (*c*) $S = \frac{1}{4}(6\frac{3}{4}) = 1\frac{11}{16}$.

6. $S = 2ws + rt$. (*a*) $S = 3 + 4 = 7$; (*b*) $S = 8 + 7\frac{1}{2} = 15\frac{1}{2}$; (*c*) $S = 10 + 9 = 19$.

7. The square on $(a - b)$ equals the square on a plus the small square on b minus two rectangles whose dimensions are a and b; or $(a - b)^2 = a^2 + b^2 - 2ab$.

8. The rectangle whose dimensions are $(a + b)$ and $(a - b)$ equals the square on a minus the square on b, since the two small rectangles are equal; or $(a + b)(a - b) = a^2 - b^2$.

9. The rectangle whose dimensions are $(a + c)$ and $(a + b)$ equals the square on a plus three rectangles whose dimensions are a and b, a and c, and b and c; or $(a + c)(a + b) = a^2 + ab + ac + bc = a^2 + a\ (b + c) + bc$.

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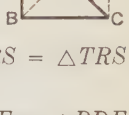
Supplementary Exercises. Areas of Triangles

10. ABC is an isosceles right triangle (Given); draw $AD \parallel BC$ and $CD \parallel BA$ (*Prin.* 14); $ABCD$ is a square (§ 141); $ABCD = \frac{(AC)^2}{2}$ (§ 249); $\triangle ABC = \frac{1}{2}\, ABCD$ (§ 146); $\triangle ABC = \frac{1}{2} \cdot \frac{c^2}{2} = \frac{c^2}{4}$ (Sub.).



11. $PRST$ is a $\square$ (Given); $PT \parallel RS$ (§ 147); $\triangle PRS = \triangle TRS$ (§ 246).

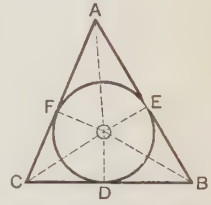
12. $CE = EB$ (Given); $\triangle ACE = \triangle ABE$ and $\triangle CDE = \triangle BDE$ (§ 245); $\triangle ADC = \triangle ABD$ (*Prin.* 6).

13. $S = a^2 - 4(\frac{1}{2}\, ab) = a^2 - 2\, ab = a\,(a - 2\, b)$ (§§ 250 and 242)

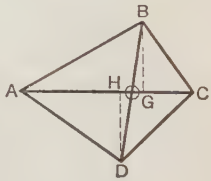
14. $ABCD$ is a $\square$ (Given); draw $AG \perp BD$ and $CH \perp BD$ (*Prin.* 15); rt. $\triangle AGD \cong$ rt. $\triangle CHB$ (*hy. a.* = *hy. a.*); $AG = CH$ (*Prin.* 38); $\triangle 1 = \triangle 2$ and $\triangle 3 = \triangle 4$ (§ 245).

15. $\frac{1}{2} \cdot 3\, x \cdot 4\, x = 150$; $6\, x^2 = 150$; $x^2 = 25$; $x = 5$; $5\, x =$ the hypotenuse $= 25$; $\frac{1}{2} \cdot 25\, h = 150$; $h = 12$.

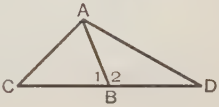
16. Let D, E, and F be the points of tangency of the inscribed circle with the triangle. Draw OA, OB, OC, OD, OE, and OF (*Prin.* 13); $OD \perp BC$, $OE \perp AB$, $OF \perp AC$ (§ 208); $\triangle OCB = \frac{1}{2}\, OD \times CB$, $\triangle OBA = \frac{1}{2}\, OE \times AB$, $\triangle OAC = \frac{1}{2}\, OF \times AC$ (§ 242); $OD = OE = OF = r$ (*Prin.* 33); $\triangle ABC = \frac{1}{2}\, r\,(AB + BC + CA) = \frac{1}{2}\, pr$ (*Prin.* 5 and Sub.).



17. $\triangle ABC = \triangle ACD$ (Given); draw altitudes BG and DH (*Prin.* 15); $\frac{\triangle ABC}{\triangle ACD} = \frac{BG}{DH}$ (§ 244); $\frac{BG}{DH} = \frac{\triangle ABC}{\triangle ACD} = 1$ (*Prin* 2); $BG = DH$ (*Prin.* 2); $\angle BOG = \angle DOH$ (*Prin.* 27); rt. $\triangle BGO \cong$ rt. $\triangle DHO$ (*a.l.* = *a.l.*); $BO = OD$ (*Prin.* 38).



18. $AB = A'B'$ and $BC = B'D$ (Given); $\angle 1 + \angle 2 =$ st. $\angle$ (Given); place the trian-



gles as shown so that $A'B'$ coincides with AB (*Prin.* 37); CBD is a st. line (*Prin.* 30); $\triangle ABC$ and $\triangle ABD$ have a common altitude from A (*Prin.* 15); $\triangle ABC = \triangle ABD$ (§ 245).

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Supplementary Exercises. Areas of Parallelograms

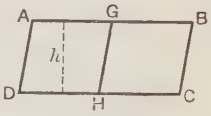
19. Rhombus $(R) = \frac{1}{2}\, dd'$ (§ 248); $R = \frac{1}{2}\, d \times 10 = 100$; $5\, d = 100$; $d = 20$.

20. Let s = side of original square; $d = s\sqrt{2}$ (§ 251); $d^2 = 2\, s^2$ (*Prin.* 8).

21. $R = S$; $R = \frac{1}{2} \times 18 \times 4 = 36$; $S = s^2 = 36$; $s = 6$.

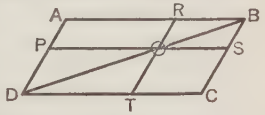
22. $BCDE$ is a $\square$ (Given); $DG = GE$ (Given); $\angle A = \angle B$ and $\angle D = \angle E$ (§ 101); $\triangle ADG \cong \triangle BGE$ (*s.a.a.* = *s.a.a.*); $BCDG = BCDG$ (Iden.); $\triangle ABC = BCDE$ (*Prin.* 5).

23. $ABCD$ is a $\square$ (Given); $AG = GB = \frac{1}{2}\, AB$ and $DH = HC = \frac{1}{2}\, DC$ (Given); $AB = DC$ (§ 147); $DH = HC = AG = GB$ (*Prin.* 8 and Sub.); $AB \parallel DC$ (§ 138); $AGHD$ and $GBCH$ are [S] (§ 152, *a*); let h be the common perpendicular to AB and DC; $AGHD = GBCH$ (§ 256).



24. $ABCD$ is a $\square$ (Given); produce CB (*Prin.* 11); draw AG and $DH \perp BC$ (*Prin.* 15); $AB = CD$ (§ 147); $AB \parallel CD$ (§ 138); $\angle B = \angle C$ (§ 102); rt. $\triangle 1 \cong$ rt. $\triangle 2$ (*hy. a.* = *hy. a.*); $AGCD = AGCD$ (Iden.); $AGCD - \triangle 1 = AGCD - \triangle 2$ (*Prin.* 6); $\square ABCD$ = rectangle $AGHD$ (Sub.); $AGHD = AD \times DH$ (*Prin.* 39); $\square ABCD = AD \times DH$ (Sub.).

25. $ABCD$ is a $\square$ (Given); $PS \parallel AB \parallel DC$ and $RT \parallel AD \parallel BC$ (Given); $PARO$, $BSOR$, $POTD$, and $OSCT$ are [S] (§ 138); $\triangle ABD \cong \triangle BCD$, $\triangle DOP \cong \triangle DOT$, and $\triangle BOR \cong \triangle BOS$ (§ 146); $\triangle DOP + \triangle BOR = \triangle DOT + \triangle BOS$ (*Prin.* 5); $PARO = OSCT$ (*Prin.* 6).



26. $3\, x = 5\, y$; $x - y = 8$; $3\,(y + 8) = 5\, y$; $y = 12$; $x = 20$; $S = \frac{1}{2} \times 12 \times 20 = 120$.

27. $AB \parallel DC$ (§ 138); $OB =$ OD (Given); $\angle OBG = \angle ODH$

and $\angle OGB = \angle OHD$ (§ 101); $\triangle BOG \cong \triangle DOH$ (*s.a.a.* = *s.a.a.*); $\triangle BDA \cong \triangle BDC$ (§ 146); $DAGO = CHOB$ (*Prin.* 6); $DAGH = BCHG$ (*Prin.* 5).

28. $AB \parallel XY$, $AC \parallel AX$, $BC \parallel YZ$, $AX \parallel BY \parallel CZ$ (Given); $ABYX$, $ACZX$, and $BCZY$ are ▱ (§ 138); $AB = XY$, $AC = XZ$, and $BC = YZ$ (§ 147); $\triangle ABC \cong \triangle XYZ$ (*s.s.s.* = *s.s.s.*); $BCZXY = BCZXY$ (Iden.); $\square AZ + \square AY = \square BZ$ (*Prin.* 6).

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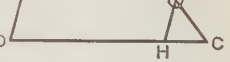
Supplementary Exercises. Areas of Trapezoids

29. $S = \frac{1}{2} h (b + b')$; (*a*) $x = \frac{1}{2} \times 6 (8 + 10) = 54$; (*b*) $x = \frac{1}{2} \times 10 (7 + 13) = 100$; (*c*) $42 = \frac{1}{2} x (6 + 8)$; $x = 6$; (*d*) $88 = \frac{1}{2} \cdot 8 (x + 12)$; $x = 10$; (*e*) $90 = \frac{1}{2} \cdot 9 (6 + x)$; $x = 14$.

30. $S = \frac{1}{2} h (b + b')$. (*a*) $240 = \frac{1}{2} \times 16 (b + b + 3)$; $b = 13.5$; $b + 3 = 16.5$; (*b*) $240 = \frac{1}{2} \times 16 (2b + 3b)$; $2b = 12$; $3b = 18$.

31. Call the upper and the lower figures from left to right respectively *A*, *B*, *C*, *D*, and *E*: $\triangle A = \frac{1}{2} \times 12 \times 21 = 126$; trapezoid $B = \frac{1}{2} \times 22 (21 + 16) = 407$; $\triangle C = \frac{1}{2} \times 10 \times 16 = 80$; $\triangle D = \frac{1}{2} \times 20 \times 26 = 260$; $\triangle E = \frac{1}{2} \times 20 \times 18 = 180$; polygon = 1053.

32. $AB \parallel CD$ (Given); $BO = OC$ (Given); $\angle B = \angle C$ and $\angle G = \angle H$ (§ 101); $\triangle BOG \cong \triangle HOC$ (*s.a.a.* = *s.a.a.*); $ABOHD = ABOHD$ (Iden.); $DAGH = ABCD$ (*Prin.* 5).



33. $ABCD$ is a trapezoid with median GH (Given); $GO = OH$ (Given); $GH \parallel AB \parallel DC$ (§ 156); $OP = OR$ (§ 282); GO and OH are medians (§ 157); $APRD = h \times GO$ and $PBCR = h \times OH$ (§ 258); $APRD = PBCR$ (Sub.).

34. Draw a line connecting two points on the figure, the longer the line, the better; draw perpendiculars to the line at equal intervals; connect the consecutive end points to form two triangles and a number of trapezoids; the approximate area of the figure is the sum of the areas of the triangles and the trapezoids.

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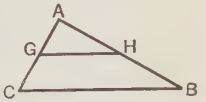
Supplementary Exercises. Ratios of Line Segments

35. $MN \parallel BC$ and $MO \parallel AC$ (Given); $BO : OC = x : y$ and $CN : NA = x : y$ (§ 279); $BO : OC = CN : NA$ (Sub.).

36. $OA : AC = OP : PS$ and $OB : BC = OR : RS$ (Given); $AP \parallel CS$ and $BR \parallel CS$ (§ 284); $AP \parallel BR$ (§ 118); $OB : OA = OR : OP$ (§ 280).

37. $AG = GC$ and $AH = HB$ (Given); $\frac{AG}{GC} = \frac{AH}{HB} = 1$ (*Prin.* 2); $GH \parallel CB$ (§ 284).

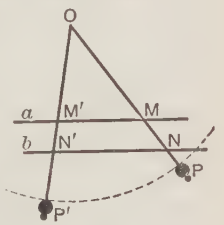


38. A line from a vertex of a triangle which divides the opposite side externally into segments which have the same ratio as the other two sides bisects an exterior angle at that vertex. Use the figure of § 287; draw $\perp s$ g and h (*Prin.* 15); $\frac{\triangle OAC}{\triangle OAB} = \frac{OC}{OB}$ (§ 244); $\frac{\triangle OAC}{\triangle OAB} = \frac{h \times AC}{g \times AB}$ (§ 243); $\frac{OC}{OB} = \frac{h \times AC}{g \times AB}$ (Sub.); $\frac{OC}{OB} = \frac{AC}{AB}$ (Given); $1 = \frac{h}{g}$ (*Prin.* 8); $h = g$ (*Prin.* 2); OA is the bisector of an exterior angle (§ 173).

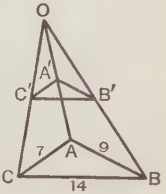
39. $GH \parallel AB \parallel DC$ (Given); $AG : GD = BH : HC$ (§ 281).

40. $DE \parallel BC$ (Given); draw $EF \parallel AB$ (*Prin.* 14); $AE : AC = BF = BC$ (§ 280); $BF = DE$ (§ 148); $AE : AC = DE : BC$ (Sub.)

41. $a \parallel b$ (Given); let OP' represent any other position of the pendulum different from OP and which cuts the $\parallel$s at M' and N'; $OM : ON = OM' : ON'$ = a constant (§ 280).



42. $OA' = A'A, OB' = B'B$, etc. (Given); $A'B' = \frac{1}{2} AB$ (§ 155); $AB = 9$ (Given); $A'B' = \frac{1}{2} \times 9 = 4\frac{1}{2}$ and likewise $A'C' = \frac{1}{2} \times 7 = 3\frac{1}{2}$ and $B'C' = \frac{1}{2} \times 14 = 7$ (Sub.); $A'B' + B'C' + A'C' = 4\frac{1}{2} + 3\frac{1}{2} + 7 = 15$.



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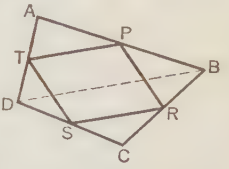
Miscellaneous Exercises

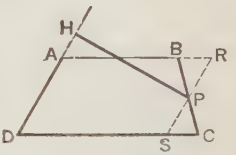
1. Draw DB (*Prin.* 13); $PT \parallel BD$ and $RS \parallel BD$ (§ 155); $PT \parallel RS$ (§ 118); $PT = \frac{1}{2} BD$ and $RS = \frac{1}{2} BD$ (§ 155); $PT = RS$ (Sub.); $PRST$ is a $\square$ (§ 152); $\frac{\triangle APT}{\triangle ABD} = \frac{\frac{1}{2} AD \times \frac{1}{2} AB}{AD \times AB} = \frac{1}{4}$ (§ 261) and likewise $\frac{\triangle CRS}{\triangle CBD} = \frac{1}{4}$; $\triangle ABD = 4\ \triangle APT$ and $\triangle CBD = 4\ \triangle CRS$ (*Prin.* 7); $ABCD = 4(\triangle APT + \triangle CRS)$ (*Prin.* 5) and likewise $ABCD = 4(\triangle BRP + \triangle DST)$; $2\ ABCD = 4(\triangle APT + \triangle CRS + \triangle BRP + \triangle DST)$ (*Prin.* 5); $\triangle APT + \triangle CRS + \triangle BRP + \triangle DST = \frac{1}{2} ABCD$ or $PRST = \frac{1}{2} ABCD$ (*Prin.* 8).



2. Let a, b, and c be the sides and p, r, and s the corresponding altitudes of the triangle; $\triangle abc = \frac{1}{2} ap = \frac{1}{2} br = \frac{1}{2} cs$ (§ 242); $ap = br = cs$ (*Prin.* 7); $p = r = s$ (Given); $a = b = c$ (*Prin.* 8); $\triangle abc$ is equilateral (§ 61).

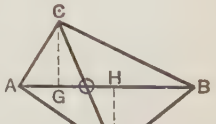
3. $CE \parallel BD$ (Given); $\triangle BDC = \triangle BDE$ (§ 246); $\triangle ABD = \triangle ABD$ (Iden.); $ABCD = \triangle ABE$ (*Prin.* 5).

4. Draw $RS \parallel AD$ (*Prin.* 14); produce AB to meet RS (*Prin.* 11); $ARSD$ is a $\square$ (§ 138); $ARSD = PH \times AD$ (§ 253); $ABCD = ARSD$ (p. 201, Ex. 32); $ABCD = ARSD = PH \times AD$ (Sub.).



5. In $\triangle T$ and $\triangle T'$, $\angle 1 = \angle 2$, a and b include $\angle 1$, c and d include $\angle 2$, $\triangle T = \triangle T'$ (Given); $\frac{\triangle T}{\triangle T'} = \frac{ab}{cd}$ (§ 261); $\frac{\triangle T}{\triangle T'} = 1$ (*Prin.* 2); $\frac{ab}{cd} = 1$ (Sub.); $ab = cd$ (*Prin.* 2).

6. $\triangle ABC = \triangle ABD$ (Given); draw altitudes CG and DH (*Prin.* 15); $\frac{\triangle ABC}{\triangle ABD} = \frac{CG}{DH}$ (§ 244); $\frac{\triangle ABC}{\triangle ABD} = 1$ (*Prin.* 2); $\frac{CG}{DH} = 1$ (Sub.); $CG = DH$ (*Prin.* 2); rt. $\triangle CGO \cong$ rt. $\triangle DHO$ (*a.l.* = *a.l.*); $OC = OD$ (*Prin.* 38).



7. Place the triangles as in the figure (*Prin.* 37); $\angle 1 + \angle 2 =$ st. $\angle$ (Given); DCB is a st. line (*Prin.* 30); draw AD (*Prin.* 13); $\frac{\triangle DEC}{\triangle DAC} = \frac{CE}{CA}$ and $\frac{\triangle DAC}{\triangle ABC} = \frac{DC}{CB}$ (§ 241); $\frac{\triangle DEC}{\triangle ABC} = \frac{CE \times CD}{CA \times CB}$ (*Prin.* 7).

8. $OP \parallel DA$ and $OR \parallel DC$ (Given); $\frac{BP}{PA} = \frac{BO}{OD}$ and $\frac{BO}{OD} = \frac{BR}{RC}$ (§ 279); $\frac{BP}{PA} = \frac{BR}{RC}$ (Sub.); $PR \parallel AC$ (§ 284).

9. $S = \frac{1}{2} bh$; $h = \frac{1}{3} b$; (*a*) $24 = \frac{1}{2} b \cdot \frac{b}{3}$; $b = 12$; $h = 4$; (*b*) $54 = \frac{1}{2} b \cdot \frac{b}{3}$; $b = 18$; $h = 6$; (*c*) $150 = \frac{1}{2} b \cdot \frac{b}{3}$; $b = 30$; $h = 10$.

10. $AD = DC$, $AE = EB$ (Given); draw DE (*Prin.* 13); $DE \parallel CB$ (§ 155); $\triangle DCB = \triangle ECB$ (§ 246); $\triangle BOC = \triangle BOC$ (Iden.); $\triangle DOC = \triangle EOB$ (*Prin.* 6); $\triangle BDC = \triangle BDA$ (§ 245); $\triangle BCO = ADOE$ (*Prin.* 6).

11. $ABCD$ is a trapezoid (Given); $AB \parallel CD$ (§ 137); $\triangle ABD = \triangle ABC$ and $\triangle ADC = \triangle BDC$ (§ 246); $\triangle OCD = \triangle OCD$ (Iden.); $\triangle OAD = \triangle OBC$ (*Prin.* 6).

12. $\frac{a}{b} = \frac{c}{d}$ and $\frac{p}{r} = \frac{s}{t}$ (Given); $x \parallel y$ and $z \parallel y$ (§ 284); $x \parallel z$ (*Prin.* 16).

13. $\triangle 1 = \triangle 2$ (Given); $\triangle AOB = \triangle AOB$ (Iden.); $\triangle ABD = \triangle ABC$ (*Prin.* 5); $AB \parallel CD$ (§ 247).

14. Draw AP (*Prin.* 13); $AR = RC$ and $AS = SB$ (Given); $\triangle ARP = \triangle CRP$ and $\triangle APS = \triangle BPS$ (§ 245); $ASPR = \triangle CRP + \triangle BPS = \frac{1}{2} \triangle ABC$ (*Prin.* 5).

CHAPTER FOUR

PAGE 208, § 292

Suggestion: Although pupils who have advanced thus far in geometry should realize that *definitions are reversible* (cf. § 39), as teachers of experience know, pupils frequently fail to appreciate when once polygons (especially triangles) have been established as similar that certain pairs of angles are equal and also that certain ratios of sides are equal. The direct use of § 292 appears in only two of the following propositions (§§ 293 and 305), whereas the reversed sense of the definition appears time and time again in propositions and exercises. **Note and emphasize the sentence in black-face type on p. 208.**

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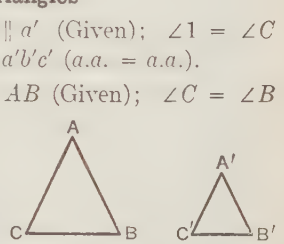
Exercises. Similar Triangles

1. Use the first figure of § 300: $a \parallel a'$ (Given); $\angle 1 = \angle C$ (§ 102); $\angle A = \angle A$ (Iden.); $\triangle abc \sim \angle a'b'c'$ (*a.a.* = *a.a.*).

2. GH and $PR \perp CB$ (Given); $AC = AB$ (Given); $\angle C = \angle B$ (§ 80); rt. $\triangle CHG \sim$ rt. $\triangle BRP$ (§ 295).

3. $AC = AB$ and $A'C' = A'B'$ (Given); $\angle C' = \angle C$ (Given); $\angle C' = \angle B'$ and $\angle C = \angle B$ (§ 80); $\angle B' = \angle B$ (Sub.); $\triangle ABC \sim \triangle A'B'C'$ (*a.a.* = *a.a.*).



4. $\angle P = \angle P$ (Iden.); $\angle C = \angle D$ (§ 227); $\triangle PCA \sim \triangle PDB$ (*a.a.* = *a.a.*).

5. $AE \perp BC$ and $BD \perp AC$ (Given); $\angle DAO = \angle CAE$ (Iden.); $\triangle AEC \sim \triangle ADO$ (§ 295).

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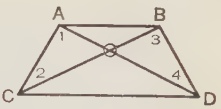
Exercises. Similar Triangles

1. $BC : B'C' = AC : A'C'$ (Given); $\angle C = \angle C'$ (*Prin.* 26); $\triangle ABC \sim \triangle A'B'C'$ (§ 300).

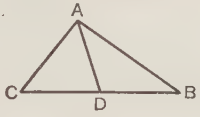
2. $AB \parallel CD$ (Given); $\angle BAO = \angle ODC$ and $\angle ABO = \angle OCD$ (§ 101); $\triangle AOB \sim \triangle DOC$ (*a.a.* = *a.a.*).

3. $AB \parallel CD$ (Given); $\angle ABO = \angle OCD$ and $\angle BAO = \angle ODC$ (§ 101); $\triangle AOB \sim \triangle DOC$ (*a.a.* = *a.a.*); draw $AM \perp CD$ and $BN \perp CD$ (*Prin.* 15); $AM = BN$ (§ 148); rt. $\triangle AMC \cong$ rt. $\triangle BND$ (*hy.l.* = *hy.l.*); $\angle C = \angle D$ (*Prin.* 38); $\frac{AC}{CD} = \frac{BD}{CD}$ (*Prin.* 8); $\triangle ACD \sim \triangle BCD$ and in like manner $\triangle ABC \sim \triangle ABD$ (§ 300); $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$ (§ 292); $\triangle AOC \sim BOD$ (*a.a.* = *a.a.*).



4. $PA = AS$ and $PB = BR$ (Given); $AB \parallel RS$ (§ 155); $\angle BAR = \angle ARS$ and $\angle ABS = \angle BSR$ (§ 101); $\triangle AOB \sim \triangle ROS$ (*a.a.* = *a.a.*); $\angle P = \angle P$ (Iden.); $\angle PAB = \angle PSR$ (§ 102); $\triangle PAB \sim \triangle PSR$ (*a.a.* = *a.a.*).

5. $\triangle ABC \sim \triangle A'B'C'$ (Given); $\angle C = \angle C'$ (§ 292); $AC : A'C' = CB : C'B'$ (§ 292); $CD = \frac{1}{2} CB$ and $C'D' = \frac{1}{2} C'B'$ (Given); $CD : C'D' = CB : C'B'$ (*Prin.* 8); $AC : A'C' = CD : C'D'$ (Sub.); $\triangle ACD \sim \triangle A'C'D'$ (§ 300) and in like manner $\triangle ADB \sim \triangle A'D'B'$.



6. $\angle B = \angle B$ (Iden.); $BH : BC = BC : BD$ (Given); $\triangle BCH \sim \triangle BCD$ (§ 300).

7. $\angle B = \angle B$ (Iden.); $\angle BCH = \angle D$ (Given); $\triangle BCH \sim \triangle BDC$ (*a.a.* = *a.a.*).

8. $GD \perp CH$ (Given); $\angle 1 = \angle 2$ (*Prin.* 26); $CD : GD = GD : DH$ (Given); $\triangle CDG \sim \triangle GDH$ (§ 300).

9. $CG \perp GH$ and $GD \perp CH$ (Given); $\angle C = \angle C$ (Iden.); rt. $\triangle CDG \sim$ rt. $\triangle CGH$ (§ 295).

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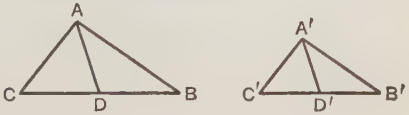
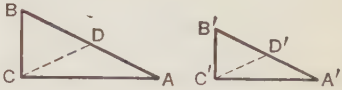
Exercises. Similar Triangles

1. $\frac{AC}{A'C'} = \frac{CB}{C'B'} = \frac{AD}{A'D'}$ (Given); $CD = \frac{1}{2} CB$ and $C'D' = \frac{1}{2} C'B'$ (Given);



$CD : C'D' = CB : C'B'$ (*Prin.* 8); $\frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'}$ (Sub.); $\triangle ACD \sim \triangle A'C'D'$ (§ 301). $\angle C = \angle C'$ (§ 292); $\triangle ABC \sim \triangle A'B'C'$ (§ 300).

2. In rt. $\triangle ABC$ and rt. $\triangle A'B'C'$, $AB : A'B' = BC : B'C'$ (Given); draw medians CD and $C'D'$ (*Prin.* 13); $CD = BD = \frac{1}{2} AB$ and $C'D' = B'D' = \frac{1}{2} A'B'$ (Ex. 15, p. 120); $\frac{CD}{C'D'} = \frac{BD}{B'D'} = \frac{AB}{A'B'}$ (*Prin.* 8); $\frac{BC}{B'C'} = \left(\frac{AB}{A'B'}\right) = \frac{BD}{B'D'} = \frac{CD}{C'D'}$ (Sub.); $\triangle BCD \sim \triangle B'C'D'$ (§ 301); $\angle B = \angle B'$ (§ 292); rt. $\triangle ABC \sim$ rt. $\triangle A'B'C'$ (§ 295).



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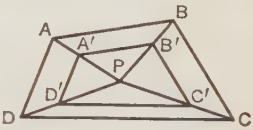
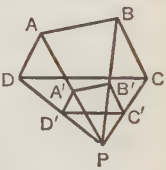
Exercises. Similar Polygons

1. $A'B' \parallel AB$, $B'C' \parallel BC$, and $D'C' \parallel DC$ (Given); $\frac{PA'}{A'A} = \frac{PB'}{B'B} = \frac{PC'}{C'C} = \frac{PD'}{D'D}$ (§ 279, Sub.); $A'D' \parallel AD$ (§ 284); in $\triangle PA'B'$ and $\triangle PAB$, $\angle PA'B' = \angle PAB$ and $\angle PB'A' = \angle PBA$ (§ 102); $\triangle PB'A' \sim \triangle PBA$ and likewise $\triangle PB'C' \sim \triangle PBC$, etc. (*a.a.* = *a.a.*); $ABCD \sim A'B'C'D'$ (§ 305).



2. $A'B' \parallel AB$, $B'C' \parallel BC$, and $D'C' \parallel DC$ (Given); $\frac{PA'}{A'A} = \frac{PB'}{B'B} = \frac{PC'}{C'C} = \frac{PD'}{D'D}$ (§279, Sub.); $A'D' \parallel AD$ (§ 284); $\angle DAB = \angle D'A'B$, $\angle ABC = \angle A'B'C'$ (§ 134); $\angle PAB = \angle PA'B'$ and $\angle PBA = \angle PB'A'$ (§ 102); $\triangle PAB \sim \triangle PA'B'$, $\triangle PBC \sim \triangle PB'C'$, etc. (§ 294); $\frac{AB}{A'B'} = \left(\frac{PB}{PB'}\right) = \frac{BC}{B'C'} = \left(\frac{PC}{PC'}\right) = \frac{CD}{C'D'}$ = etc. (§ 292); $ABCD \sim A'B'C'D'$ (§ 292).



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Exercises. Intersecting Chords

1. $pr = st$; (*a*) $3p = 24$; $p = 8$; (*b*) $7s = 84$; $s = 12$; (*c*) $8t = 48$; $t = 6$.

2. (*a*) $x(7 - x) = 6 \times 2; 7x - x^2 = 12; -x^2 + 7x - 12 = 0;$ $x^2 - 7x + 12 = 0;$ $(x - 4)(x - 3) = 0;$ $x = 4;$ $7 - x = 3;$ (*b*) $x = 6;$ $9 - x = 3;$ (*c*) $x = 4;$ $10 - x = 6.$

3. (*a*) $x^2 = 4 \times 9 = 36;$ $x = 6;$ (*b*) $x^2 = 16 \times 25 = 400;$ $x = 20.$

4. (*a*) $x^2 = 4 \times 16 = 64;$ $x = 8;$ (*b*) $x^2 = 9 \times 25 = 225;$ $x = 15.$

5. $xy = 10 \times 8 = 80;$ yes.

6. See the first five statements of the proof of § 313.

7. The longest chord is a diameter of 30″; the shortest chord is perpendicular to the diameter; $x \cdot x = 27 \cdot 3 = 81;$ $x = 9;$ $2x = 18.$

8. $x \cdot x = 2 \cdot 18 = 36;$ $x = 6;$ $2x = 12.$

9. $18x = 24 \cdot 24 = 576;$ $x = 32;$ diameter $= 18 + 32 = 50;$ radius $= 25.$

10. The diameter $= 5 + x;$ $5x = 10 \cdot 10 = 100;$ $x = 20;$ diameter $= 5 + x = 5 + 20 = 25;$ $\frac{c}{2} \cdot \frac{c}{2} = 9 \times 16;$ $\frac{c^2}{4} = 144;$ $c^2 = 576;$ $c = 24.$

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Exercises. Tangents and Secants

1. $s : t = t : e;$ $t^2 = se;$ (*a*) $t^2 = 9 \times 4 = 36;$ $t = 6;$ (*b*) $t^2 = 25 \times 4 = 100;$ $t = 10;$ (*c*) $t^2 = 25 \times 9 = 225;$ $t = 15.$

2. (*a*) $27e = 9 \times 9 = 81;$ $e = 3;$ (*b*) $16e = 64;$ $e = 4;$ (*c*) $20e = 100;$ $e = 5.$

3. $s = i + e;$ $t^2 = se;$ (*a*) $t^2 = 4 \times 9 = 36;$ $t = 6;$ (*b*) $t^2 = 9 \times 25 = 225;$ $t = 15;$ (*c*) $t^2 = 4 \times 16 = 64;$ $t = 8.$

4. $e(i + e) = t^2;$ (*a*) $e(9 + e) = 36;$ $e^2 + 9e = 36;$ $e^2 + 9e - 36 = 0;$ $(e + 12)(e - 3) = 0;$ $e = 3;$ $s = 9 + 3 = 12;$ (*b*) $e(7 + e) = 144;$ $e^2 + 7e = 144;$ $e^2 + 7e - 144 = 0;$ $(e + 16)(e - 9) = 0;$ $e = 9;$ $s = 7 + 9 = 16.$

5. Let t be a tangent from O to the circle; $OB : t = t : OA$ and $OC : t = t : OD$ (§ 314); $OA \times OB = t^2$ and $OC \times OD = t^2$ (§ 266); $OA \times OB = OC \times OD$ (Sub.); $OB : OC = OD : OA$ (§ 268).

6. $OB : OC = OD : OA$ (Ex. 5); $OB \times OA = OC \times OD$ (§ 266).

7. $\angle O = \angle O$ (Iden.); $OB : OC = OD : OA$ (Ex. 5); $\triangle OCB \sim \triangle OAD$ (§ 300).

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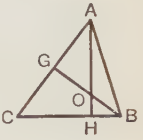
Exercises. Proportions and Products from Similar Triangles

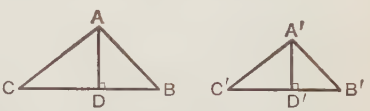
1. $x : 8 = 50 : 10$; $x = 40$.



2. $AH \perp BC$ and $BG \perp AC$ (Given); $\angle AOG = \angle BOH$ (*Prin.* 27); rt. $\triangle AOG \sim$ rt. $\triangle BOH$ (§ 295); $OA : OB = AG : BH$ (§ 292); $OA \times BH = OB \times AG$ (§ 266).

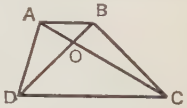
3. $AH \perp BC$ and $BG \perp AC$ (Given); $\angle C = \angle C$ (Iden.); rt. $\triangle ACH \sim$ rt. $\triangle BCG$ (§ 295); $AC : BC = CH : CG$ (§ 292).

4. $\triangle ABC \sim \triangle A'B'C'$ (Given); $AD \perp BC$ and $A'D' \perp B'C'$ (Given); $\angle C = \angle C'$ (§ 292); rt. $\triangle ACD \sim$ rt. $\triangle A'C'D'$ (§ 295);



$AD : A'D' = AC : A'C'$ (§ 292); $\dfrac{AC}{A'C'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ (§ 292); $\dfrac{AD}{A'D'} = \dfrac{AC}{A'C'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ (Sub.).

5. $\triangle OAB \sim \triangle OCD$; $OB : OD = OA : OC$; $9 : 3 = x : 16 - x$; $x = 12$; $16 - x = 4$.



6. $\angle ADB = \angle ACE$ (Given); $\angle B = \angle E$ (§ 227); $\triangle ADB \sim \triangle ACE$ (*a.a.* = *a.a.*); $AB : AE = AD : AC$ (§ 292); $AB \times AC = AD \times AE$ (§ 266).

7. $6 : 10 = 8 : x$; $x = 13\frac{1}{3}$; $6 : 10 = 12 : y$; $y = 20$.

8. Use Ex. 5, p. 212, and § 292.

9. Draw BD (*Prin.* 13); $\angle BAE = \angle BAD$ (Iden.); $AB = AC$ (Given); $\overset{\frown}{AB} = \overset{\frown}{AC}$ (§ 196); $\angle ABC = \angle ADB$ (§ 227); $\triangle BAE \sim \triangle BAD$ (*a.a.* = *a.a.*); $AD : AB = AB : AE$ (§ 292).

10. $\angle CAE = \angle CAD$ (Iden.); $AB = AC$ (Given); $\overset{\frown}{AB} = \overset{\frown}{AC}$ (§196); $\angle ADC = \angle ACE$ (§ 227); $\triangle CAE \sim \triangle CAD$ (*a.a.* = *a.a.*); $AD : AC = AC : AE$ (§ 292).

11. $\angle GOB =$ rt. $\angle$ (§ 27); $\angle H =$ rt. $\angle$ (§ 226); $\angle GOB = \angle H$

(*Prin.* 26); $\angle OBG = \angle HBA$ (Iden.); $\triangle GOB \sim \triangle AHB$ (*a.a.* = *a.a.*); $AB : BG = BH : OB$ (§ 292); $AB \times OB = BH \times BG$ (§ 266).

12. See the proof of § 313.

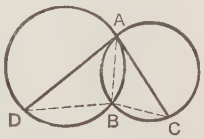
13. Draw BD (*Prin.* 13); AD is a diameter (Given); $\angle B = 90°$ (§ 226); $EC \perp AD$ (Given); $\angle A = \angle A$ (Iden.); rt. $\triangle ACE \sim$ rt. $\triangle ABD$ (§ 295); $AD : AE = AB : AC$ (§ 292); $AC \times AD = AB \times AE$ (§ 266).

14. E and D are mid-points (Given); $ED \parallel CB$ (§ 155); $\angle AED = \angle C$ (§ 102); $\angle A = \angle A$ (Iden.); $\triangle AED \sim \triangle ACB$ (*a.a.* = *a.a.*); $\frac{ED}{CB} = \frac{AE}{AC}$ (§ 292); $\frac{AE}{AC} = \frac{1}{2}$ (Given); $\frac{ED}{CB} = \frac{AE}{AC} = \frac{1}{2}$ (Sub.).

15. Use proof of Ex. 2, p. 212, and $OA : OD = OB : OC$ (§ 292).

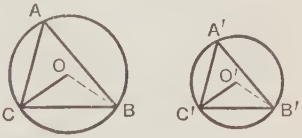
16. $15 : 10 = 9 : x$; $x = 6$; $9 - x = 3$; $15 : 10 = 12 : y$; $y = 8$; $12 - y = 4$.

17. Draw BA, BC, and BD (*Prin.* 13); AD and AC are tangents (Given); $\angle D = \angle CAB$ and $\angle C = \angle DAB$ (§ 228); $\triangle ABC \sim \triangle ABD$ (*a.a.* = *a.a.*); $BC : AB = AB : BD$ (§ 292).



18. $\overset{\frown}{AC} = \overset{\frown}{BC}$ (Given); draw AC and AD (*Prin.* 13); in $\triangle CAE$ and $\triangle CAD$, $\angle ACE = \angle ACD$ (Iden.); $\angle D = \angle CAE$ (§ 227); $\triangle ACE \sim \angle ACD$ (*a.a.* = *a.a.*); $CE : AC = AC : CD$ (§ 292).

19. $\triangle ABC \sim \triangle A'B'C'$ (Given); OB, OC, $O'B'$, and $O'C'$ are radii of the circumscribed circles (Given); $\angle A = \angle A'$ (§ 292);



$\angle A =$ numerically $\frac{1}{2}\overset{\frown}{CB}$ (§ 224); $\angle O =$ numerically $\overset{\frown}{CB}$ (*Prin.* 31); $2\ \angle A =$ numerically $\overset{\frown}{CB}$ (*Prin.* 7); $\angle O = 2\ \angle A$ (Sub.) and likewise $\angle O' = 2\ \angle A'$; $\angle O = \angle O'$ (*Prin.* 7); $OB = OC$ and $O'B' = O'C'$ (*Prin.* 33); $\frac{OB}{O'B'} = \frac{OC}{O'C'}$ (*Prin.* 8); $\triangle OBC \sim \triangle O'B'C'$ (§ 300); $\frac{OB}{O'B'} = \frac{BC}{B'C'}$ (§ 292).

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Exercises. Right Triangles

1. Use the fig. of § 320; a, b, and h are mean proportionals; $c : a = a : e$; $c : b = b : d$; $d : h = h : e$.

2. $c : a = a : e$; $75 : 60 = 60 : e$; $e = 48$; $d = c - e = 75 - 48 = 27$; $d : h = h : e$; $27 : h = h : 48$; $h = 36$.

3. $d : h = h : e$; $45 : h = h : 80$; $h = 60$; $c = d + e = 125$; $c : a = a : e$; $125 : a = a : 80$; $a = 100$; $c : b = b : d$; $125 : b = b : 45$; $b = 75$.

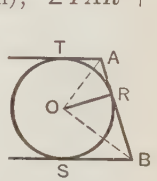
4. $c : b = b : d$; $c : 60 = 60 : 36$; $c = 100$; $e = c - d = 100 - 36 = 64$; $c : a = a : e$; $100 : a = a : 64$; $a = 80$.

5. $e : h = h : d$; $112 : 84 = 84 : d$; $d = 63$; $c = d + e = 63 + 112 = 175$; $c : a = a : e$; $175 : a = a : 112$; $a = 140$; $c : b = b : d$; $175 : b = b : 63$; $b = 105$.

6. $\triangle abc = \frac{1}{2} ab = \frac{1}{2} hc$; $c : b = b : d$; $c : 135 = 135 : 81$; $c = 225$; $e = 225 - 81 = 144$; $d : h = h : e$; $81 : h = h : 144$; $h = 108$; $c : a = a : e$; $225 : a = a : 144$; $a = 180$; $\triangle abc = \frac{1}{2} \times 180 \times 135 = \frac{1}{2} \times 108 \times 225 = 12{,}150$.

7. Diameter: $AD = AD : CD$ (§ 320); $D : 12 = 12 : 8$; $D = 18$; $R = 9$.

8. Draw OA and OB (*Prin.* 13); $TA \parallel SB$ (Given); $\angle TAR + \angle SBR = 180°$ (§ 104); $\angle OAR = \frac{1}{2} \angle TAR$ and $\angle OBR = \frac{1}{2} \angle SBR$ (§ 211); $\frac{1}{2} \angle TAR + \frac{1}{2} \angle SBR = 90°$ (*Prin.* 8); $\angle OAR + \angle OBR = 90°$ (Sub.); $\angle O + \angle OAR + \angle OBR = 180°$ (§ 106); $\angle O = 90°$ (*Prin.* 6); $OR \perp AB$ (§ 208); $AR : OR = OR : RB$ (§ 317).



9. Use the fig. of Ex. 6; $a = 2b$ (Given); $a \perp b$ and $h \perp c$ (Given); $c : a = a : e$ and $c : b = b : d$ (§ 319); $a^2 = ce$ and $b^2 = cd$ (§ 266); $\frac{a^2}{b^2} = \frac{ce}{cd} = \frac{e}{d}$ (*Prin.* 8); $a^2 = 4b^2$ (*Prin.* 9); $\frac{4b^2}{b^2} = \frac{e}{d} = \frac{4}{1}$ (Sub.).

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Exercises. The Pythagorean Theorem

2. (*a*) $c = 30$; (*b*) $c = 39$; (*c*) $c = 50$.

4. $b = 28$.

5. $a = 21$.

6. $s = 6\sqrt{2} = 8.484$.

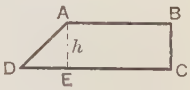
7. (*a*) 17; (*b*) 61; (*c*) 85.

8. $a = 24$.

9. (*a*) 13; (*b*) $4^2 + 15^2 = c^2$; $c = \sqrt{241} = 15.5+$.

10. $c = 37$; the height of the tree $= 35 + 37 = 72$.

11. Draw $AE \perp DC$; $\angle D = 45° = \angle DAE$; $AE = DE = h$; $h^2 + h^2 = 8^2$; $2h^2 = 64$; $h^2 = 32$; $h = 4\sqrt{2}$; $AB = EC = 12$; $DC = 12 + 4\sqrt{2}$; area $= \frac{1}{2} \cdot 4\sqrt{2}\,(12 + 4\sqrt{2}) = 24\sqrt{2} + 16 = 49.936$.

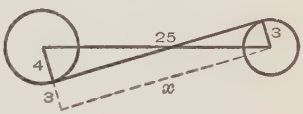


12. Draw r; $5^2 + 12^2 = r^2$; $25 + 144 = r^2 = 169$; $r = 13$; $x^2 + 5^2 = 13^2$; $x = 12$.

13. $5^2 + b^2 = 13^2$; $b = 12$.

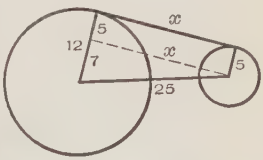
14. $b^2 + 20^2 = 25^2$; $b = 15$; (*a*) $19^2 + x^2 = 25^2$; $361 + x^2 = 625$; $x^2 = 264$; $x = 16.248$; distance the foot moves $= 16.248 - 15 = 1.248$; (*b*) $y^2 + 16^2 = 25^2$; $y^2 + 256 = 625$; $y^2 = 369$; $y = 19.209$; distance the top moves $= 20 - 19.209 = .791$.

15. $125^2 + 125^2 = c^2$; $c = 125\sqrt{2} =$ 176.75 feet south from the treasure.



16. $x^2 + 7^2 = 25^2$; $x^2 + 49 = 625$; $x^2 = 576$; $x = 24$.

17. $x^2 + 7^2 = 25^2$; $x^2 + 49 = 625$; $x^2 = 576$; $x = 24$.



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Exercises. Area of a Triangle by Hero's Formula

2. (*a*) $s = 24$; $S = \sqrt{24 \times 14 \times 7 \times 3} = \sqrt{(8 \times 3)\,(7 \times 2)\,(7)\,(3)} = 3 \times 7 \times 4 = 84$; (*b*) $s = 56$; $S = 420$; (*c*) $s = 42$; $S = 336$; (*d*) $s = 240$; $S = 2{,}640$.

4. (*a*) $s = 27$; $S = 126$; $S = \frac{1}{2}\,bh$; $126 = \frac{1}{2} \times 21 \times h$; $h = 12$; (*b*) $s = 34$; $S = 204$; $204 = \frac{1}{2} \times 17 \times h$; $h = 24$; (*c*) $s = 52$; $S = 468$; $468 = \frac{1}{2} \times 39 \times h$; $h = 24$.

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Exercises. The Equilateral Triangle

1. $h = \frac{s}{2}\sqrt{3} = 2\sqrt{3};\ 3\sqrt{3};\ 4\sqrt{3};\ 2.5\sqrt{3};\ 3.5\sqrt{3}.$

2. $S = \left(\frac{s}{2}\right)^2\sqrt{3} = 4\sqrt{3};\ 9\sqrt{3};\ 16\sqrt{3};\ 6\frac{1}{4}\sqrt{3}.$

3. (*a*) $h = \frac{s}{2}\sqrt{3}; 10\sqrt{3} = \frac{s}{2}\sqrt{3};\ s = 20;$ (*b*) $16\sqrt{3} = \frac{s}{2}\sqrt{3}; s = 32;$ (*c*) $4 = \frac{s}{2}\sqrt{3};\ s = \frac{8}{\sqrt{3}} = \frac{8}{3}\sqrt{3};$ (*d*) $6 = \frac{s}{2}\sqrt{3}; s = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$

4. (*a*) $S = \left(\frac{s}{2}\right)^2\sqrt{3};\quad 25\sqrt{3} = \left(\frac{s}{2}\right)^2\sqrt{3},\ 5 = \frac{s}{2};\quad s = 10;$ (*b*) $36\sqrt{3} = \left(\frac{s}{2}\right)^2\sqrt{3};\quad s = 12;\quad$ (*c*) $173.2 = \left(\frac{s}{2}\right)^2 \times 1.732;$ $100 = \left(\frac{s}{2}\right)^2;\ s = 20.$

5. $h = \frac{s}{2}\sqrt{3};\ s = \frac{2h}{\sqrt{3}} = \frac{2h}{3}\sqrt{3}.\ S = \frac{s^2}{4}\sqrt{3};\ s^2 = \frac{4S}{\sqrt{3}} = \frac{4S\sqrt{3}}{3};$ $s = 2\sqrt{\frac{S\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3S\sqrt{3}}.$

6. The area of a hexagon $(S) = \left(\frac{s}{2}\right)^2\sqrt{3} \times 6 = 16\sqrt{3} \times 6 = 96\sqrt{3}.$

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Exercises. The 45-45-90 Triangle

1. $4\sqrt{2};\ 10;\ 6;\ 3\sqrt{2}.$

2. $8;\ 32;\ 16;\ 8\sqrt{2}.$

3. $\frac{\sqrt{2}}{1} = 1.414;\ \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = .707;\ \frac{1}{1} = 1.000.$

4. $CD = 10;\ AC = BC = 10\sqrt{2};\ AB = 20;\ AD = DB = 10.$

5. 45-45-90 △; it is the same.

6. $p = 5\sqrt{2}.$

7. Altitude: base = 1 : 2.

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Exercises. The 30-60-90 Triangle

1. (*a*) $AC = 3$; $BC = 3\sqrt{3}$; (*b*) $AC = \frac{8}{3}\sqrt{3}$; $AB = \frac{16}{3}\sqrt{3}$; (*c*) $BC = 12$; $AB = 8\sqrt{3}$; (*d*) $AB = 24$; $BC = 12\sqrt{3}$.

2. (*a*) $S = \frac{1}{2} \times 10 \times 10\sqrt{3} = 50\sqrt{3}$; (*b*) $S = \frac{1}{2} \times 10\sqrt{3} \times 30 = 150\sqrt{3}$; (*c*) $S = \frac{1}{2} \times 6\sqrt{3} \times 6 = 18\sqrt{3}$; (*d*) $S = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$; (*e*) $S = \frac{1}{2} \times 4 \times 4\sqrt{3} = 8\sqrt{3}$.

3. $\frac{BC}{AB} = \frac{s\sqrt{3}}{2s} = \frac{\sqrt{3}}{2} = .866$; $\frac{AC}{AB} = \frac{s}{2s} = \frac{1}{2} = .500$; $\frac{BC}{AC} = \frac{s\sqrt{3}}{s} = \frac{\sqrt{3}}{1} = 1.732$.

4. 5; $5\sqrt{3}$.

5. $\angle DBC = 45°$; $\angle ABC = 60°$; $\angle ABD = 15°$; $\angle ADB = 135°$; $AB = 20$; $AC = 10\sqrt{3} = 17.32$; $DC = 10$; $AD = 17.32 - 10 = 7.32$; $BD = 10\sqrt{2} = 14.14$.

6. $AC = 10$; $f = \frac{10}{3}\sqrt{3}$; $d = e = \frac{20}{3}\sqrt{3}$.

7. By taking $AB = s$; then $AC = \frac{s}{2}$; $BC = \frac{s}{2}\sqrt{3}$.

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Exercises

1. $\sin A = \frac{BC}{AB} = \cos B$.

2. $\cos A = \frac{AC}{AB} = \sin B$.

3. $\sin B' = \frac{AC'}{AB'} = \cos A$.

4. $\angle B = \angle B'$; $\sin B = \sin B'$.

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Exercises. Trigonometric Ratios

1. $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = .707$; $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = .707$; $\tan 45° = \frac{1}{1} = 1.000$.

2. $\sin 30° = \frac{1}{2} = .500$; $\cos 30° = \frac{\sqrt{3}}{2} = .866$; $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = .577$.

3. $\sin 60° = \frac{\sqrt{3}}{2} = .866$; $\cos 60° = \frac{1}{2} = .500$; $\tan 60° = \frac{\sqrt{3}}{1} = 1.732$.

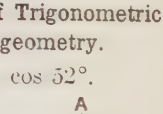
4. $\sin A = \frac{5}{13} = .384$; $A = 22°+$; $\sin B = \frac{12}{13} = .923$; $B = 67°+$.

Note: Should the teacher so desire, the values of the angles of this and other exercises may be more closely approximated by interpolation. For example: $\sin A = \frac{5}{13} = .3846$. From the table, since .3846 lies between .3746 and .3907, the value of $\angle A$ lies between 22° and 23°. (.3907 − .3746 = .0161 and .3846 − .3746 = .0100.) One degree of difference may be considered as divided into 161 equal parts. We wish to find the fractional part of a degree to which 100 of these parts correspond. That is, $\angle A$ is $\frac{100}{161}$ of 1° more than 22° = $22\frac{100}{161}$ degrees = 22° 37′.

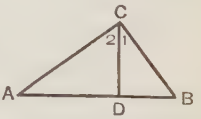
The teacher is cautioned, however, against spending too much time on this phase of the work, for **the entire topic of Trigonometric Ratios is not required by leading syllabi for plane geometry.**

5. .4067 = sin 24° = cos 66°; .6157 = sin 38° = cos 52°.

6. Draw $CD \perp AB$; $\sin A = \frac{CD}{AC}$; $\sin 80° = \frac{CD}{20}$; $CD = 20 \times .9848 = 19.696$; $S = \frac{1}{2} AB \times CD = \frac{1}{2} \times 20 \times 19.696 = 196.96$.

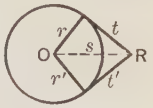


7. $\sin B = \frac{CD}{CB} = \frac{12}{15} = .800$; $\angle B = \angle 2 = 53°+$; $\angle A = \angle 1 = 37°-$.



8. Use the fig. of Ex. 9; let $\angle DAB = 30°$ and $\angle DAC = 40°$; $\cos 30° = \frac{AD}{AB}$; $.866 = \frac{8}{AB}$; $AB = 9.2$; $\tan 30° = \frac{BD}{AD}$; $.5774 = \frac{BD}{8}$; $BD = 4.619$; $\cos 40° = \frac{AD}{AC}$; $.766 = \frac{8}{AC}$; $AC = 10.4$; $\tan 40° = \frac{CD}{AD}$; $.839 = \frac{CD}{8}$; $CD = 6.712$; $BC = BD + DC = 4.619 + 6.712 = 11.331$.

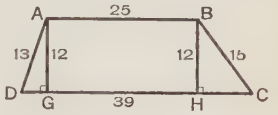
9. $\sin 50° = \frac{AD}{AB}$; $.766 = \frac{10}{AB}$; $AB = 13.05$; $\tan 50° = \frac{AD}{BD}$; $1.19 = \frac{10}{BD}$; $BD = 8.4$; $\sin 30° = \frac{AD}{AC}$; $.5 = \frac{10}{AC}$; $AC = 20$; $\tan 30° = \frac{AD}{DC}$; $.577 = \frac{10}{DC}$; $DC = 17.3$; $BC = BD + DC = 8.4 + 17.3 = 25.7$.



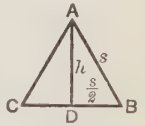
10. $\tan 39° = \frac{r}{t}$; $.8098 = \frac{r}{15}$; $r = 12.147$.

11. Let O be the center of the circle with chord AB; draw $OH \perp AB$; $\angle O = 72°$; $\angle AOH = 36°$; $\sin 36° = \frac{AH}{OA}$; $.5878 = \frac{AH}{12}$; $AH = 7.0536$; $AB = 2\,AH = 14.1072$.

12. $\sin D = \frac{12}{13} = .923$; $D = 67°+$; $\sin C = \frac{12}{15} = .800$; $C = 53°+$; $\angle A = 180° - 67°+ = 113°-$; $\angle B = 180° - 53°+ = 127°-$.



13. $\angle C = 60°$; $p = 48$; $s = 16$; $\sin 60° = \frac{h}{AC}$; $.866 = \frac{h}{16}$; $h = 13.856$; $S = \frac{1}{2}bh = \frac{1}{2} \times 16 \times 13.856 = 110.848$.



Optional solution, which may be used as a check: $S = \left(\frac{s}{2}\right)^2 \sqrt{3} = 64 \times 1.732 = 110.848$.

14. $\tan 62° = \frac{BC}{AB}$; $1.8807 = \frac{BC}{350}$; $BC = 658.245$.

15. $\tan A = \frac{h}{2\,h} = \frac{1}{2} = .500$; $\angle A = 26°+$.

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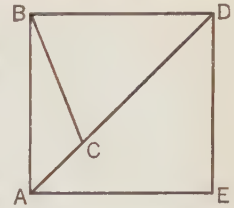
Exercises. Drawing to Scale

1. $CD = \frac{9}{16}''$; $1''$ represents $96''$; $\frac{9}{16}''$ represents $54''$; $BD = CD = 54''$.

2. $2''$ represents $16'$; $3\frac{1}{4}''$ represents $26'$; $\frac{3}{4}''$ represents $6'$; $\frac{3}{16}''$ represents $1\frac{1}{2}'$.

3. 8′ is represented by 1″; 60′ is represented by $7\frac{1}{2}$″; 168′ is represented by 21″; 18′ is represented by $2\frac{1}{4}$″; 36″ = 3′ is represented by $\frac{3}{8}$″.



4. Let $AB = \frac{5}{16}$″ = 150 mi. and $AC = \frac{6}{16}$″ = 60 mi. with $\angle BAC = 45°$; then $\frac{1}{16}$″ represents 10 mi; $BC = \frac{23}{32}$″ = 115 mi.

5. Draw a triangle ABC making BC = 3″, letting 1″ = 100′. The approximate measure of the line AB will be found to be $3\frac{9}{16}$″ = 3.5625″. That is, AB represents 356.25 ft. By calculation, $\tan 50° = \frac{AB}{300}$ or AB = 357.54 ft. The error is 1.29 ft. or less than $\frac{1}{3}$ of 1%. In finding the distance across a river, $\angle B$ must be a right angle; the distance BC, $\angle C$, and the scale used may vary; the method is indicated by what has already been given.

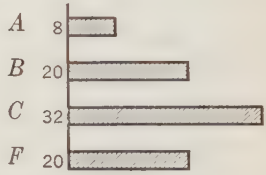
6. Let 16″ = $\frac{1}{4}$″; 80″ = $1\frac{1}{4}$″; 192″ = 3″; 208″ = $3\frac{1}{4}$″.

7. 1″ = 320 mi.; $\frac{1}{32}$″ = 10 mi.; Austin to Dallas = $\frac{21}{32}$″ = 210 mi.; Austin to Houston = $\frac{17}{32}$″ = 170 mi.; Austin to San Antonio = $\frac{7}{32}$″ = 70 mi.; Austin to El Paso = $1\frac{25}{32}$″ = 570 mi.

8. Length of living room = 26′ 8″ = 320″ = $\frac{53}{32}$″; $\frac{1}{32}$″ = 14−″; width of living room = 16′ 8″ = 200″ = $\frac{14}{32}$″; $\frac{1}{32}$″ = 14+″.

9. Length of the solarium = $\frac{22}{32}$″ = 308″ = 25′ 8″; width of the solarium = $\frac{11}{32}$″ = 154″ = 12′ 10″.

10. Let $\frac{1}{32}$″ represent 1 pupil;
A pupils = $\frac{8}{32}$″.
B pupils = $\frac{20}{32}$″.
C pupils = $\frac{32}{32}$″.
Failures = $\frac{20}{32}$″.



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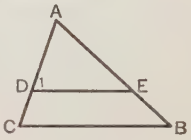
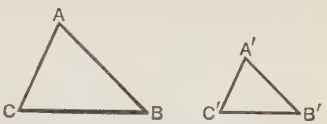
Supplementary Exercises. Similar Triangles

1. $AD = DC$, $AE = EB$ (Given); $DE \parallel CB$ (§ 155); $\angle 1 = \angle C$ (§ 102); $\angle A = \angle A$ (Iden.); $\triangle ADE \sim \triangle ACB$ (*a.a.* = *a.a.*).



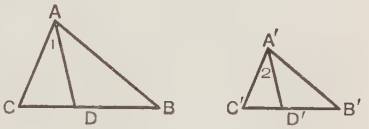
2. $AC = AB$, $A'C' = A'B'$, $\angle A = \angle A'$ (Given); $\angle C = \angle B$, $\angle C' = \angle B'$ (§ 80); $\angle C + \angle B = \angle C' + \angle B'$ (§ 107, Sub.); $2\ \angle C = 2\ \angle C'$ (Sub.); $\angle C = \angle C'$ (*Prin.* 8); $\triangle ABC \sim \triangle A'B'C'$ (*a.a.* = *a.a.*).



3. $\angle DAC = \angle DBC$, $\angle ADB = \angle ACB$, $\angle CAB = \angle CDB$, $\angle DBA = \angle DCA$ (§ 227); $\angle P = \angle P$ (Iden.); $\angle C + \angle DAB = 180°$ (Ex. 6, p. 152); $\angle BAP + \angle DAB = 180°$ (*Prin.* 29); $\angle C = \angle BAP$ (*Prin.* 28); $\triangle DOA \sim \triangle BOC$, $\triangle AOB \sim \triangle DOC$, $\triangle PDB \sim \triangle PCA$, $\triangle PAB \sim \triangle PDC$ (*a.a.* = *a.a.*).

4. $AE \perp CB$ and $BD \perp AC$ (Given); $\angle DAO + \angle C =$ rt. $\angle$ and $\angle DBC + \angle C =$ rt. $\angle$ (§ 108); $\angle DAO = \angle DBC$ (*Prin.* 28); rt. $\triangle AOD \sim$ rt. $\triangle BCD$ (§ 295).

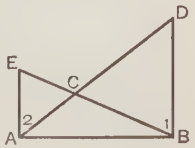
5. $\triangle ABC \sim \triangle A'B'C'$ (Given); $\angle C = \angle C'$ and $\angle A = \angle A'$ (§ 292); $\angle 1 = \frac{1}{2}\ \angle A$ and $\angle 2 = \frac{1}{2}\ \angle A'$ (Given); $\angle 1 = \angle 2$ (*Prin.* 8); $\triangle ACD \sim \triangle A'C'D'$ and likewise $\triangle ABD \sim \triangle A'B'D'$ (*a.a.* = *a.a.*).

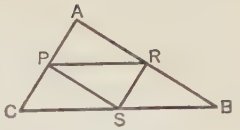


6. $\angle E = \angle E$ (Iden); $ABCD$ is a $\square$ (Given); $AD \parallel BC$ and $AB \parallel DC$ (§ 138); $\angle EBF = \angle A$ (§ 102); $\triangle EBF \sim \triangle EAD$ (*a.a.* = *a.a.*); $\angle E = \angle CDF$ and $\angle EBF = \angle C$ (§ 101); $\triangle EBF \sim \triangle CDF$ (*a.a.* = *a.a.*); $\triangle EAD \sim \triangle CDF$ (§ 298).

7. $PG = GS$, $PH = HR$ (Given); $GH \parallel SR$ (§ 155); $\angle HGO = \angle ORS$, $\angle GHO = \angle OSR$ (§ 101); $\triangle GOH \sim \triangle ROS$ (*a.a.* = *a.a.*); $\angle PGH = \angle PSR$, $\angle PHG = \angle PRS$ (§ 102); $\triangle PGH \sim \triangle PSR$ (*a.a.* = *a.a.*); $\angle S = \angle R$ (§ 80); $\triangle GSR \cong \triangle HSR$ (*s.a.s.* = *s.a.s.*); $\angle HSR = \angle GRS$ (*Prin.* 38); $\angle GSO = \angle HRO$ (*Prin.* 6); $\angle GOS = \angle HOR$ (*Prin.* 27); $\triangle GOS \sim \triangle HOR$ (*a.a.* = *a.a.*).

8. AE and BD are $\perp AB$ (Given); $AE \parallel BD$ (§ 98); $\angle E = \angle 1$ and $\angle 2 = \angle D$ (§ 101); $\triangle ACE \sim \triangle BCD$ (*a.a.* = *a.a.*).





9. P, R, and S are the mid-points of the sides of $\triangle ABC$ (Given); $PR \parallel CB$, $RS \parallel AC$, and $PS \parallel AB$ (§ 155); $\triangle ABC \sim \triangle PRS$ (§ 296).

10. $\angle 1 = \angle 2$ (Given); $\angle ACD = \angle AEB$ (§ 227); $\triangle ACD \sim \triangle AEB$ (*a.a.* = *a.a.*); $\angle CEA = \angle DBA$ (§ 227); $\triangle AEC \sim \triangle ABD$ (*a.a.* = *a.a.*); $\angle 1 = \angle DBE$ (§ 227); $\triangle ADC \sim BDE$ (*a.a.* = *a.a.*).

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Supplementary Exercises. Similar Triangles

11. $PZ = ZC$, $PX = XA$ (Given); $ZX \parallel CA$ (§ 155); $\angle PZX = \angle PCA$, $\angle PXZ = \angle PAC$ (§ 102); $\triangle PXY \sim \triangle PAC$ and likewise $\triangle PYZ \sim \triangle PBC$ and $\triangle PXY \sim \triangle PAB$ (*a.a.* = *a.a.*); $\triangle XYZ \sim \triangle ABC$ (§ 296).

12. Let A and B be the points of intersection of RO with KH and TS, C the intersection of RT and KH, and D of RS and KH. There are at least twelve pairs of triangles which can be proved similar by *a.a.* = *a.a.* They are ORS and OGH, ORT and OGK, OST and OHK, OBT and OAK, OBS and OAH, RST and GHK, GKA and RTB, GAH and RBS, RAC and RBT, RAD and RBS, GKA and RAC, GAH and RAD.

13. $TN = NF$, $TL = LD$, $TM = ME$ (Given); $LN \parallel DF$, $LM \parallel DE$ (§ 155); $\angle TNL = \angle TFD$, $\angle TLN = \angle TDF$, $\angle TLM \angle TDE$, $\angle TML = \angle TED$ (§ 102); $\triangle TNL \sim \triangle TFD$, $\triangle TLM \sim \triangle TDE$, $\triangle LMN \sim \triangle DEF$ (*a.a.* = *a.a.*).

14. $AC \perp DB$ (Given); $DO = OB = 2\,OA = \frac{1}{2}\,OC$ (Given); $\frac{OA}{OB} = \frac{1}{2} = \frac{OB}{OC}$ (*Prin.* 8 and Sub.); rt. $\angle AOB$ = rt. $\angle BOC$ (*Prin.* 26); $\triangle BOA \sim \triangle BOC$ (§ 300); $\frac{OA}{OB} = \frac{1}{2} = \frac{OD}{OC}$ (*Prin.* 8 and Sub.); rt. $\angle AOB$ = rt. $\angle COD$ (*Prin.* 26); $\triangle BOA \sim \triangle COD$ (§ 300); $\frac{OA}{OB} = \frac{1}{2} = \frac{OA}{OD}$ (*Prin.* 8 and Sub.); rt. $\angle DOA$ = rt. $\angle BOA$ (*Prin.* 26); $\triangle DOA \sim \triangle BOA$ (§ 300).

15. $ABCD$ is a square (Given); $AN = NM = MD = \frac{1}{3}\,AD$ and $AE = EF = FB = \frac{1}{3}\,AB$ (Given); $AD = AB$ (§ 141); $\frac{1}{3}\,AD =$

$\frac{1}{3}\,AB$ (*Prin.* 7); $AN = AE$ (Sub.); $\angle A$ = rt. $\angle$ (§ 141); $\angle ANE + \angle AEN$ = rt. $\angle$ (§ 108); $\angle ANE = \angle AEN$ (§ 80); $2\ \angle ANE = 90°$ (Sub.); $\angle ANE = 45°$. There are six such right triangles with a 45° angle, including $\triangle AEN$ and $\triangle CGL$. All are similar by § 295.

16. $\angle D = \angle D$ (Iden.); $AE = 2\,ED = 4\,BC = \frac{4}{3}\,CD$ (Given); $\frac{1}{4}\,AE = \frac{1}{2}\,ED = BC = \frac{1}{3}\,CD$ (*Prin.* 8); let $BC = x$; $CD = 3\,x$, $ED = 2\,x$, and $AE = 4\,x$ (*Prin.* 7); $AD = 6\,x$ and $BD = 4\,x$ (*Prin.* 5); $\frac{DE}{DC} = \frac{2\,x}{3\,x} = \frac{2}{3}$ and $\frac{BD}{AD} = \frac{4\,x}{6\,x} = \frac{2}{3}$ (*Prin.* 8); $\frac{DE}{DC} = \frac{BD}{AD}$ (Sub.); $\triangle ADB \sim \triangle CDE$ (§ 300).

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Supplementary Exercises. Intersecting Chords

17. Let the chords from left to right be $2\,a$, $2\,b$, $2\,c$, $2\,d$, and $2\,e$; $a \times a = 2 \times 10$; $a^2 = 20$; $a = 2\sqrt{5}$; $2\,a = 4\sqrt{5} = 8.944$; $b \times b = 4 \times 6 = 24$; $b = 2\sqrt{6}$; $2\,b = 4\sqrt{6}$; $2\,b = 9.796$; $2\,c$ = diameter = 12; $2\,d = 2\,b = 9.796$ and $2\,e = 2\,a = 8.944$ (§ 203).

18. Let r = the radius; $(r + 5)(r - 5) = 8 \times 18$; $r^2 - 25 = 144$; $r^2 = 169$; $r = 13$; $d = 26$.

19. Let d = the diameter; $4(d - 4) = 12 \times 12$; $4\,d - 16 = 144$; $d = 40$.

20. $h\,(D - h) = \frac{c}{2} \cdot \frac{c}{2}$; $Dh - h^2 = \frac{c^2}{4}$; $Dh = \frac{c^2}{4} + h^2 = \frac{c^2 + 4\,h^2}{4}$; $D = \frac{c^2 + 4\,h^2}{4\,h}$.

21. $D = \frac{c^2 + 4\,h^2}{4\,h} = \frac{36^2 + 4 \times 4^2}{4^2} = 9^2 + 4 = 85$.

22. Let r = the radius; $(r + 4)(r - 4) = 3 \times 3$; $r^2 - 16 = 9$; $r^2 = 25$; $r = 5$; $d = 10$.

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Supplementary Exercises. Proportions and Products from Similar Triangles

23. $AB = AC$ (Given); $BC = CD$ (Given); $\angle B = \angle BCA$ and $\angle B = \angle BDC$ (§ 80); $\angle BCA = \angle BDC$ (Sub.); $\angle B = \angle B$ (Iden.);

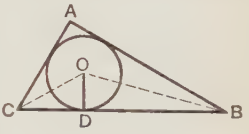
$\triangle ABC \sim \triangle BCD$ (*a.a.* = *a.a.*); $AB : BC = BC : BD$ (§ 292); $(BC)^2 = BD \times BA$ (§ 266).

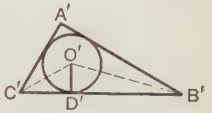
24. Draw OA and BC intersecting at P (*Prin.* 13); $OB \perp BA$ (§ 208); $\angle BDC = 90°$ (Given); $AB = AC$ (§ 211); $OB = OC$ (*Prin.* 33); $OA \perp BC$ (§ 171); $\angle OBP = \angle OAB$ (*Prin.* 28); rt. $\triangle BDC \sim$ rt. $\triangle OBA$ (§ 295); $CD : OB = BD : AB$ (§ 292); $CD : BD = OB : AB$ (§ 272).

25. Draw CE (*Prin.* 13); $AD \perp DB$ (Given); $AC \perp CE$ (§ 226); $\angle E = \angle B$ (§ 227); rt. $\triangle ACE \sim$ rt. $\triangle ADB$ (§ 295); $AB : AE = AD : AC$ (§ 292); $AB : AD = AE : AC$ (§ 272).

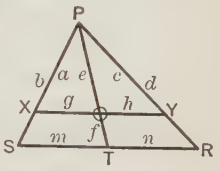
26. Draw BE (*Prin.* 13); $DE \perp AB$ and $AD = DB$ (Given); $AE = EB$ (§ 168); $\angle A = \angle ABE$ (§ 80); $AB = BC$ (Given); $\angle A = \angle C$ (§ 80); $\angle ABE = \angle A = \angle C$ (Sub.); $\triangle ABE \sim \triangle ABC$ (*a.a.* = *a.a.*); $AC : AB = AB : AE$ (§ 292).

27. $\triangle ABC \sim \triangle A'B'C'$ (Given); $\angle OBD = \frac{1}{2}\ \angle B$ and $\angle O'B'D' = \frac{1}{2}\ \angle B'$ (§ 211); $\angle B = \angle B'$ (§ 292); $\angle OBD = \angle O'B'D'$ (*Prin.* 8); $OD \perp CB$ and $O'D' \perp C'B'$ (§ 208); rt. $\triangle ODB \sim$ rt. $\triangle O'D'B'$ (§ 295); $\dfrac{OD}{O'D'} = \dfrac{OB}{O'B'}$ (§ 292); $\angle OCD = \angle O'C'D'$ (*Prin.* 8); $\triangle BOC \sim \triangle B'O'C'$ (*a.a.* = *a.a.*); $\dfrac{OB}{O'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'} = \dfrac{AB}{A'B'}$ (§ 292); $\dfrac{OD}{O'D'} = \left(\dfrac{OB}{O'B'}\right) = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'} = \dfrac{AB}{A'B'}$ (Sub.).





28. $XY \parallel SR$ (Given); $\angle PXO = \angle PST$, $\angle PYO = \angle PRT$, $\angle POX = \angle PTS$, $\angle POY = \angle PTR$ (§ 102); $\triangle POX \sim \triangle PTS$, $\triangle POY \sim \triangle PTR$, and $\triangle PXY \sim \triangle PSR$ (*a.a.* = *a.a.*); $\dfrac{a}{b} = \dfrac{g}{m}$, $\dfrac{c}{d} = \dfrac{h}{n}$, and $\dfrac{a}{b} = \dfrac{c}{d}$ (§ 292); $\dfrac{g}{m} = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{h}{n}$ (Sub.); $m = n$ (Given); $g = h$ (§ 269).



29. $BA \perp AC$ (Given); $DEFG$ is a square (Given); $DE \perp BC$ and $FG \perp BC$ (§§ 141, 104); $\angle BED = \angle C$ (*Prin.* 28); rt. $\triangle BDE$

$\sim$ rt. $\triangle FGC$ (§ 295); $BD : FG = DE : GC$ (§ 292); $FG = DE$ (§ 147); $BD : DE = DE : GC$ (Sub.).

30. Draw radii OA and $O'B$ to the extremities of the secant and draw the line of centers OO' (*Prin.* 13); the line of centers passes through the point of tangency (C) of the circles (§ 215, B); $OA = OC$ and $O'C = O'B$ (*Prin.* 33); $\angle 4 = \angle 1$ and $\angle 2 = \angle 3$ (§ 80); $\angle 1 = \angle 2$ (*Prin.* 27); $\angle 4 = \angle 3$ (Sub.); $\triangle OAC \sim \triangle O'BC$ (*a.a.* = *a.a.*); $AC : BC = OA : O'B$ (§ 292).

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Supplementary Exercises. The Pythagorean Theorem

31. $(a^2 - b^2)^2 + (2\,ab)^2 = c^2$; $a^4 - 2\,a^2b^2 + b^4 + 4\;\; a^2b^2 = c^2$; $a^4 + 2\,a^2b^2 + b^4 = c^2$; $a^2 + b^2 = c$.

32. $x^2 + (x + 1)^2 = (x + 2)^2$; $x^2 + x^2 + 2\,x + 1 = x^2 + 4\,x + 4$; $x^2 - 2\,x - 3 = 0$; $(x - 3)\;\; (x + 1) = 0$; $\;x = 3$; $\;x + 1 = 4$; $x + 2 = 5$.

33. Let the segments of the diameter (35) be x and $35 - x$; $x\,(35 - x) = \frac{84}{5} \cdot \frac{84}{5}$; $35\,x - x^2 = \frac{7056}{25}$; $875\,x - 25\,x^2 = 7056$; $25\,x^2 - 875\,x + 7056 = 0$; $x = -b \pm \sqrt{\frac{b^2 - 4\,ac}{2\,a}}$;

$x = \frac{875 \pm \sqrt{765625 - 705600}}{50}$; $x = \frac{875 \pm 245}{50}$; $x = 22.4$; $35 - x = 12.6$; $a^2 = (12.6)^2 + (16.8)^2$; $a = 21$; $b^2 = (22.4)^2 + (16.8)^2$; $b = 28$.

34. Let the segments of the line of centers be a and b; $a^2 + 8^2 = 10^2$; $a = 6$; $b^2 + 8^2 = 17^2$; $b = 15$; the line of centers $a + b = 6 + 15 = 21$.

35. Let the tangent equal $2\,x$; $x^2 + 8^2 = 10^2$; $x = 6$; $2\,x = 12$.

36. $x^2 + (12)^2 = r^2$; $(x + 3)^2 + 9^2 = r^2$; $x^2 + 144 = x^2 + 6\,x + 9 + 81$; $6\,x = 54$; $x = 9$; $r = 15$.

37. $x^2 + 6^2 = r^2$; $(14 - x)^2 + 8^2 = r^2$; $x^2 + 36 = 196 - 28\,x + x^2 + 64$; $28\,x = 224$; $x = 8$; $r = 10$.

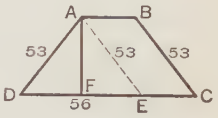
38. Let $2\,x$ = the chord; $x^2 + 6^2 = 10^2$; $x = 8$; $2\,x = 16$.

39. $(3x)^2 + (4x)^2 = (40)^2$; $9x^2 + 16x^2 = 1600$; $25x^2 = 1600$; $x^2 = 64$; $x = 8$; $3x = 24$; $4x = 32$.

40. Let x = a side; $3^2 + 4^2 = x^2$; $x = 5$; $4x = 20$; $9^2 + 12^2 = x^2$; $x = 15$; $4x = 60$.

41. Let d = the diagonal of the floor; $d^2 = (12)^2 + (16)^2$; $d = 20$; let x = the required line; $x^2 = (20)^2 + 9^2 = 481$; $x = \sqrt{481} = 21.9$.

42. Draw altitude AF. $(AF)^2 + (28)^2 = (53)^2$; $AF = \sqrt{2809 - 784} = \sqrt{2025} = 45$; the area cannot be found since the bases are unknown.



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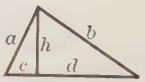
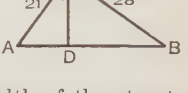
Miscellaneous Exercises

1. $a^2 = c^2 + h^2$ and $b^2 = d^2 + h^2$ (§ 321); $a^2 - b^2 = c^2 - d^2$ (*Prin.* 6).



2. (*a*) The height of the gable $= \frac{1}{2} \times 30 = 15$; (*b*) the length of the rafters $= 15\sqrt{2} = 21.2$ (§ 328).

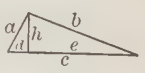
3. $(AB)^2 = (21)^2 + (28)^2$; $AB = 35$; let $AD = x$; $DB = 35 - x$; $x : 35 - x = 21 : 28 = 3 : 4$ (§ 286); $4x = 105 - 3x$; $x = 15$; $35 - x = 20$; rt. $\triangle ACB = \frac{1}{2} \times 28 \times 21 = 294$; $\triangle ACD : \triangle ACB = 3 : 7 = S : 294$ (§ 244); $S = 126$; $\triangle BCD : \triangle ACB = 4 : 7 = S' : 294$; $S' = 168$.



4. Let a and b equal the segments of the width of the street; $(32.5)^2 = (26)^2 + a^2$; $(32.5)^2 = (16.5)^2 + b^2$; $a^2 = 380.25$; $a = 19.5$; $b^2 = 784$; $b = 28$; $a + b = 47.5 = 47'\ 6''$.

5. Use the proof of § 322, with similar triangles A, B, and C in place of the squares, and with § 308 in place of § 237 as Fact 1.

6. (*a*) $d = 9$; $e = 16$; $c = 25$; $25 : a = a : 9$; $a = 15$; $25 : b = b : 16$; $b = 20$; (*b*) $d = 2.88$; $e = 5.12$; $c = 8$; $8 : a = a : 2.88$; $a = 4.8$; $8 : b = b : 5.12$; $b = 6.4$.

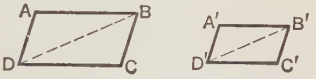


7. $6 : 9 = 8 : x$; $x = 12$; $6 : 9 = 10 : y$; $y = 15$.

8. Perimeter of the smaller triangle $= 8 + 14 + 18 = 40$; $\frac{40}{60} = \frac{8}{a} = \frac{14}{b} = \frac{18}{c} = \frac{2}{3}$; $a = 12$; $b = 21$; $c = 27$.

9. Use the figure of § 309; $P \sim P'$ (Given); $\frac{P}{P'} = \frac{a^2}{p^2}$ (§ 309); $\triangle X \sim \triangle X'$ (§ 304); $\frac{\triangle X}{\triangle X'} = \frac{a^2}{p^2} = \frac{g^2}{k^2}$ (§ 308); $\frac{P}{P'} = \frac{a^2}{p^2} = \frac{g^2}{k^2}$ (Sub.).

10. In ⑤ $ABCD$ and $A'B'C'D'$, $\angle A = \angle A'$, $\angle B = \angle B'$, etc. (Given); draw BD and $B'D'$ (*Prin.* 13); $\frac{\triangle ABD}{\triangle A'B'D'} = \frac{AB \times AD}{A'B' \times A'D'}$ (§ 261); $\triangle ABD = \frac{1}{2}\ \square\ ABCD$ and $\triangle A'B'D' = \frac{1}{2}\ \square\ A'B'C'D'$ (§ 146); $\frac{\triangle ABD}{\triangle A'B'D'} = \frac{\square\ ABCD}{\square\ A'B'C'D'}$ (*Prin.* 8); $\frac{\square\ ABCD}{\square\ A'B'C'D'} = \frac{AB \times AD}{A'B' \times A'D'}$ (Sub.).



11. Use the figures given for the solution of Ex. 5, page 212; $\triangle ACD \sim \triangle A'C'D'$ (Ex. 5, p. 212); $\frac{\triangle ACD}{\triangle A'C'D'} = \frac{(AC)^2}{(A'C')^2} = \frac{(AD)^2}{(A'D')^2}$ and $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{(AC)^2}{(A'C')^2}$ (§ 308); $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{(AC)^2}{(A'C')^2} = \frac{(AD)^2}{(A'D')^2}$ (Sub.).

12. Use the figures given for the solution of Ex. 5, p. 254; $\triangle ACD \sim \triangle A'C'D'$ (Ex. 5, p. 254); the rest of the proof is identical to that of Ex. 11.

13. $CD = DB$ or $CB = 2\ CD$; (*a*) if $\angle C = 45°$, $CD = \frac{10}{\sqrt{2}} = 5\sqrt{2}$; $CB = 10\sqrt{2}$; (*b*) if $\angle C = 60°$, $CD = 5$; $CB = 10$; (*c*) if $\angle C = 30°$, $CD = 5\sqrt{3}$; $CB = 10\sqrt{3}$.

Note: It may be interesting for a class to observe that when $\angle C$ equals 30°, 45°, and 60° in turn, CB equals $10\sqrt{3}$, $10\sqrt{2}$, and $10\sqrt{1}$.

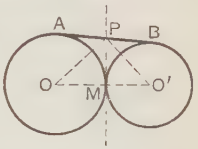
14. $\frac{75}{3} = \frac{60}{a} = \frac{50}{b} = \frac{40}{c} = 25$; $a = 2.4$; $b = 2$; $c = 1.6$; $p =$

$75 + 60 + 50 + 40 = 225$; $p' = 3 + 2.4 + 2 + 1.6 = 9$; $\frac{p}{p'} = \frac{225}{9} = 25$.

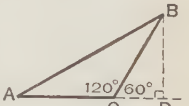
15. $\angle x = \angle x = \angle x = 45°$ (Given); $\frac{EA}{DE} = \frac{BA}{BD}$ (§ 286); the fourth acute angle at B is also 45° (§ 54); $\frac{CA}{CD} = \frac{BA}{BD}$ (§ 287); $\frac{CA}{CD} = \frac{EA}{DE}$ (Sub.).

Note: § 288 may be used if desired; $BC = BA$ (Given); $\angle C = \angle A$ (§ 80); $\triangle BCD \cong \triangle BAE$ (*a.s.a.* = *a.s.a.*); $CD = EA$ (*Prin.* 38); $\frac{CA}{EA} = \frac{EA}{DE}$ (Sub.).

16. Draw OO' (*Prin.* 13); OO' passes through M (§ 215, *B*); draw $PM \perp OO'$ at M (*Prin.* 15); $\angle APO = \angle OPM$, $\angle BPO' = \angle O'PM$ (§ 211); $\angle OPO'$ = rt. $\angle$ (§ 54, Sub., *Prin.* 8); $PM = PA = PB$ (§ 211); $2\ PM = AB$ (*Prin.* 5); $\frac{OM}{PM} = \frac{PM}{MO'}$ (§ 317); $\frac{2\ OM}{2\ PM} = \frac{2\ PM}{2\ MO'}$ (*Prin.* 7); $\frac{D}{AB} = \frac{AB}{d}$ (Sub.).



17. Produce AC (*Prin.* 11); draw $BD \perp AC$ (*Prin.* 15); $(AB)^2 = (BD)^2 + (AD)^2$ (§ 321); $AD = AC + CD$ (*Prin.* 4); $(AD)^2 = (AC)^2 + 2\ AC \times CD + (CD)^2$ (*Prin.* 9); $(BC)^2 = (BD)^2 + (CD)^2$ (§ 321); $(BD)^2 = (BC)^2 - (CD)^2$ (*Prin.* 6); $(AB)^2 = (BC)^2 - (CD)^2 + (AC)^2 + 2\ AC \times CD + (CD)^2$ (Sub.); $BC = 2\ CD$ (§ 158); $(AB)^2 = (BC)^2 + (AC)^2 + AC \times BC$ (Sub.).



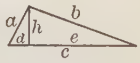
18. Let a and b represent the legs of the right triangle; $a^2 + b^2 = (50)^2 = 2{,}500$; $\frac{1}{2}\ ab = 600$; $2\ ab = 2{,}400$; $a^2 + 2\ ab + b^2 = 4{,}900$; $a + b = 70$; $a^2 - 2\ ab + b^2 = 100$; $a - b = 10$; $a = 40$; $b = 30$.

Check: $S = \frac{1}{2}\ ab = \frac{1}{2} \times 40 \times 30 = 600$.

19. $AB = AC$ (Given); draw $AH \perp CB$ (*Prin.* 15); $(AB)^2 = (AH)^2 + (BH)^2$ (§ 321); $BH = DH + BD$ (*Prin.* 4); $(BH)^2 =$

$(DH)^2 + (BD)^2 + 2\ DH \times BD$ (*Prin.* 9); $(AD)^2 = (AH)^2 + (DH)^2$ (§ 321); $(AH)^2 = (AD)^2 - (DH)^2$ (*Prin.* 6); $(AB)^2 = (AD)^2 - (DH)^2 + (DH)^2 + (BD)^2 + 2\ DH \times BD$ (Sub.); $(BD)^2 + 2\ DH \times BD = BD\ (BD + 2\ DH)$ (Factoring); $CH = HB = HD + BD$ (*Prin.* 38); $HD = HD$ (Iden.); $CD = BD + 2\ DH$ (*Prin.* 5); $(AB)^2 = (AD)^2 + CD \times DB$ (Sub.).

20. $a \perp b$ and $h \perp c$ (Given); rt. $\triangle adh \sim$ rt. $\triangle hbe$ (§ 316); $\frac{\text{rt. } \triangle adh}{\text{rt. } \triangle hbe} = \frac{a^2}{b^2}$ (§ 308).



CHAPTER FIVE

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Note: Corollary 1 has been given by some authors of geometry texts as a book proposition. It might be advisable for the teacher to require pupils to give a full proof of it. Its statement should be understood and memorized for it may be used frequently to solve exercises.

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Note: Corollaries 1–4 have many important uses in the solution of numerical problems and in the proofs of original exercises. Teachers of experience will appreciate the usefulness especially of Corollaries 2 and 4.

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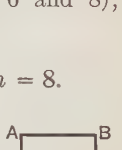
Exercises. Regular Polygons

1. Let $\angle A$ be an angle of a regular polygon; $\angle E$, its adjacent exterior angle; $\angle C$, an angle at the center of the regular polygon; $\angle A + \angle E = $ st. $\angle$ (*Prin.* 29); $\angle E = \frac{360°}{n}$ (§ 167); $\angle C = \frac{360°}{n}$ (§ 345); $\angle E = \angle C$ (Sub.); $\angle A + \angle C = $ st. $\angle$ (Sub.).

2. $\angle A = \frac{1}{2}[(360° - \overset{\frown}{MN}) - \overset{\frown}{MN}] = 180° - \overset{\frown}{MN}$, $\angle B = 180° - \overset{\frown}{NO}$, etc. (§ 230); $\angle A = \angle B = $ etc. (Given); $180° - \overset{\frown}{MN} = 180° - \overset{\frown}{NO} = $ etc. (Sub.); $\overset{\frown}{MN} = \overset{\frown}{NO} = $ etc. (*Prins.* 6 and 8); *ABCD* etc., is a regular circumscribed polygon (§ 335).

3. $\angle O = \frac{360°}{n}$ (§ 345); $n = \frac{360°}{\angle O}$; $n = 5$; $n = 12$; $n = 8$.

4. *ABCD* is a square with apothem a (Given); if $DC = s$, $DH = HC = \frac{1}{2}s$ (§ 348); draw radius r (*Prin.* 13); $\angle OCH = 45°$ (§ 348); $\angle O = 45° = \angle OCH$ (§ 108); $a = \frac{1}{2}s$ (§ 120); $s = 2a$ (*Prin.* 7).



5. (*a*) $a^2 + x^2 = (a\sqrt{2})^2 = 2a^2$; $x^2 = a^2$; $x = \pm a$; (*b*) $b^2 - \left(\frac{b}{2}\right)^2 = x^2$; $x^2 = b^2 - \frac{b^2}{4} = \frac{3b^2}{4}$; $x = \pm \frac{b\sqrt{3}}{2}$; (*c*) $x^2 = r^2 - \left(\frac{r\sqrt{2}}{2}\right)^2$; $x^2 = r^2 - \frac{r^2}{2} = \frac{r^2}{2}$; $x = \frac{r}{\sqrt{2}} = \pm \frac{r\sqrt{2}}{2}$; (*d*) $x^2 + \left(\frac{a\sqrt{3}}{2}\right)^2 = (2a)^2$; $x^2 + \frac{3a^2}{4} = 4a^2$; $x^2 = \frac{13a^2}{4}$; $x = \pm \frac{a\sqrt{13}}{2}$.

Note: The checks may easily be supplied.

6. Let $BC = s$; $\angle OBH = 30°$; $\angle O = 60°$; $a\sqrt{3} = \frac{s}{2}$; $a = \frac{s}{2\sqrt{3}} = \frac{s\sqrt{3}}{6}$.

7. $BC = s = 5$; $HB = \frac{1}{2}s = \frac{5}{2}$; $a = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}$; $r = 2a = \frac{5}{3}\sqrt{3}$.

8. *ABCDEF* is a regular polygon (Given); let a circle be circumscribed about the polygon (§ 338); $AB = BC = CD = DE =$ etc. (§ 160); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} =$ etc. (§ 196); $\overset{\frown}{BD} = \overset{\frown}{AC}$ (*Prin.* 5); $\angle 1 = \angle 2 = \angle 3$ (§ 227).

9. Use the figure for Ex. 4; $s = 4$; the diameter of the circumscribed circle $= 4\sqrt{2}$.

10. $d = 6$; $r = 3$; $s'_6 = r = 3$ (§ 346). Let *AB* be a side of the regular circumscribed hexagon; draw the apothem *a* and radius *r*; $\triangle OHB$ is a 30-60-90 triangle; $a = 3$; $HB = \frac{a}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$; $s_6 = 2\,HB = 2\sqrt{3}$.

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Exercises. Ratios of Regular Polygons

1. $\frac{s}{s'} = \frac{4}{6} = \frac{2}{3} = \frac{r}{r'} = \frac{a}{a'}$; $\frac{S}{S'} = \frac{r^2}{(r')^2} = \frac{2^2}{3^2} = \frac{4}{9}$.

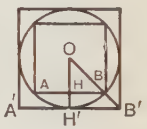
2. $\frac{S}{S'} = \frac{16}{81} = \frac{p^2}{(p')^2}$; $\frac{p}{p'} = \frac{4}{9}$.

3. Let h and h' be the apothems of the regular hexagons; $\frac{S}{S'} = \frac{h^2}{(h')^2} = \frac{(a\sqrt{3})^2}{\left(\frac{b}{2}\sqrt{3}\right)^2} = \frac{3\,a^2}{\frac{3\,b^2}{4}} = \frac{4\,a^2}{b^2}$.

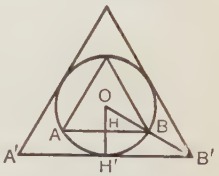
Check. $S = \frac{1}{2} \times a\sqrt{3} \times 12\,a = 6\,a^2\sqrt{3}$; $S' = \frac{1}{2} \times \frac{b}{2}\sqrt{3} \times 6\,b = \frac{6\,b^2\sqrt{3}}{4}$; $\frac{S}{S'} = \frac{4\,a^2}{b^2}$.

4. Use the figure of § 351; $\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$ (§ 351); $\angle A = \angle A'$ (Given and §§ 350, 292); $\angle 1 = \frac{1}{2}\ \angle A$ and $\angle 2 = \frac{1}{2}\ \angle A'$ (§ 348); $\angle 1 = \angle 2$ (*Prin.* 8, Sub.); in like manner $\angle OBA = \angle O'B'A'$; $\triangle OAB \sim \triangle O'A'B'$ (*a.a.* = *a.a.*); $AB : A'B' = r : r'$ (§ 292); $AB : A'B' = a : a'$ (Sub.).

5. (*a*) Let AB and $A'B'$ be sides of the inscribed and the circumscribed squares; $OB = r$; $HB = \frac{1}{2}\,r\sqrt{2}$; $AB = r\sqrt{2}$; $OH' = r$; $A'B' = 2\,r$; $\frac{AB}{A'B'} = \frac{r\sqrt{2}}{2\,r} = \frac{\sqrt{2}}{2}$.



(*b*) $OB = r$; $OH = \frac{1}{2}\,r$; $HB = \frac{1}{2}\,r\sqrt{3}$; $AB = r\sqrt{3}$; $OH' = r$; $H'B' = r\sqrt{3}$; $A'B' = 2\,r\sqrt{3}$; $\frac{AB}{A'B'} = \frac{r\sqrt{3}}{2\,r\sqrt{3}} = \frac{1}{2}$.



(*c*) $OB = r$; $HB = \frac{1}{2}\,r$; $AB = r$; $OH' = r$; $H'B' = \frac{r\sqrt{3}}{3}$; $A'B' = \frac{2\,r\sqrt{3}}{3}$; $\frac{AB}{A'B'} = \frac{r}{\frac{2\,r\sqrt{3}}{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$.

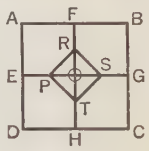
6. (*a*) $\frac{S}{S'} = \frac{(r\sqrt{2})^2}{(2\,r)^2} = \frac{2\,r^2}{4\,r^2} = \frac{1}{2}$; (*b*) $\frac{T}{T'} = \frac{(r\sqrt{3})^2}{(2\,r\sqrt{3})^2} = \frac{3\,r^2}{12\,r^2} = \frac{1}{4}$;

(*c*) $\frac{H}{H'} = \frac{r^2}{\left(\frac{2\,r\sqrt{3}}{3}\right)^2} = \frac{r^2}{\frac{4\,r^2}{3}} = \frac{3}{4}$.

Note: For pupils of a better class, the following observation may prove interesting. Let t represent the ratio of the side of the inscribed equilateral triangle to the side of the circumscribed equilateral triangle, s and h the corresponding ratios for the squares and hexagons. In like manner, let T, S, and H represent the ratios of the areas of these figures. By Ex. 5, $t : s : h = \frac{1}{2}\sqrt{1} : \frac{1}{2}\sqrt{2} : \frac{1}{2}\sqrt{3} = \sqrt{1} : \sqrt{2} : \sqrt{3}$. By Ex. 6, $T : S : H = \frac{1}{4} : \frac{2}{4} : \frac{3}{4} = 1 : 2 : 3$.

7. $S = (2r)^2 = 4r^2$; $T = \binom{s}{2}^2\sqrt{3} = 3r^2\sqrt{3}$; $\frac{S}{T} = \frac{4r^2}{3r^2\sqrt{3}} = \frac{4\sqrt{3}}{9}$.

8. Let the perimeters be respectively $3x$, $4x$, and $5x$; let the smallest and the largest of the regular polygons be P and P'; $\frac{P}{18} = \frac{(3x)^2}{(4x)^2} = \frac{9}{16}$; $P = 10\frac{1}{8}$; $\frac{P'}{18} = \frac{(5x)^2}{(4x)^2} = \frac{25}{16}$; $P' = 28\frac{1}{8}$.



9. $AB = s$; $HO = \frac{1}{2}s$; $OT = \frac{1}{4}s$; $PT = \frac{1}{4}s\sqrt{2}$; $PRST = (\frac{1}{4}s\sqrt{2})^2 = \frac{1}{8}s^2$.

PAGE 278, § 353

Suggestion: The problem may be omitted at the option of the teacher. It is not a part of the suggested Minimum Course. Its principal use in the past has been to compute an approximate value of π which is no longer required by any syllabus.

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Exercise

$S = \sqrt{r\,[2r - \sqrt{(2r - b)\,(2r + b)}]}$; $r = 4$; $b = 4\sqrt{2}$; $2r = 8$;

$S = \sqrt{4\ \ [8 - \sqrt{(8 - 4\sqrt{2})}\ \ (8 + 4\sqrt{2})]}$; $S = \sqrt{4\ \ [8 - \sqrt{32}]}$;

$S = 4\sqrt{2 - \sqrt{2}} = 3.06+$.

PAGE 280, § 358

Note: The treatment of variables and limits as given here is made to depend primarily upon substitution. Such a course is

both simple and reasonable. Substitution is one of the most important processes of mathematics; it is also one of the most satisfying.

Do not permit the loose statement that equals may be substituted for equals. Substitution is limited to an equality or an inequality. See *Prins.* 3, 40, 41, 42, and 43.

PAGE 282, STATEMENT 1

Note: Regular hexagons rather than any regular polygons of the same number of sides are inscribed in the given circles because the fact was established in § 347 that a regular hexagon can be inscribed in a circle.

PAGE 283, § 361

Suggestion: If time permits, one or more pupils may be assigned the topic "π" for a more complete account. An encyclopedia or a history of mathematics (see p. 388) may give valuable supplementary material. Ex. 13, p. 295, is interesting in this connection.

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Sight Exercises

Suggestion: In calculations involving π, it is important for pupils to learn to express all results in terms of π. If a fractional or a decimal answer is required, the evaluation should be performed as the final step.

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Sight Exercises. Circumferences and Areas of Circles

1. $C = 6\pi$; 8π; 10π; 12π.
2. $S = 9\pi$; 16π; 25π; 36π.
3. $R = 8$; 12; 16.
4. $R = 4$; 7; 10.
5. $S = 81\pi$; 121π; 400π.
6. $C = 8\pi$; 10π; 16π.

7. $1:2;\ 1:3;\ 1:2$.

8. $1:4;\ 1:9;\ 1:4$.

9. $4\pi - \pi = 3\pi$.

10. $9\pi - 4\pi = 5\pi$.

11. $4\pi - 3\pi = \pi;\ 4\pi - \frac{3}{4}\pi = \frac{7}{4}\pi$.

12. 9.

13. $\pi R^2 : R^2 = \pi : 1$.

14. $2\pi R : 4R = \pi : 2$.

15. $\frac{1}{4}\pi : \frac{1}{64}\pi = 16 : 1$.

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Suggestion: The formulas of regular polygons 1–12 may be made the basis of numerical examples, if the teacher desires. Illustrative example: If the radius of a circle is 4 (let $R = 4$), find the side and the area of an inscribed equilateral triangle. The example should be worked independently of the formulas of Exs. 5 and 11, but the results may be checked by those formulas.

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Exercises. Formulas of Regular Polygons

1, 2. See Ex. 5 (*a*), p. 277.

3, 4. See Ex. 5 (*c*), p. 277.

5, 6. See Ex. 5 (*b*), p. 277.

7. $A'_4 = (R\sqrt{2})^2 = 2R^2$.

8. $A_4 = (2R)^2 = 4R^2$.

9. $A'_6 = 6 \times \left(\frac{R}{2}\right)^2 \sqrt{3} = \frac{3R^2\sqrt{3}}{2}$.

10. $A_6 = 6 \times \left(\frac{R\sqrt{3}}{3}\right)^2 \sqrt{3} = 2R^2\sqrt{3}$.

11. $A'_3 = \left(\frac{R\sqrt{3}}{2}\right)^2 \sqrt{3} = \frac{3R^2\sqrt{3}}{4}$.

12. $A_3 = \left(\frac{2R\sqrt{3}}{2}\right)^2 \sqrt{3} = 3R^2\sqrt{3}$.

13. $\frac{A'_4}{A_4} = \frac{2\,R^2}{4\,R^2} = \frac{2}{4} = \frac{1}{2}$; $\frac{A'_6}{A_6} = \frac{3\,R^2\sqrt{3}}{2} \div 2\,R^2\sqrt{3} = 3 : 4$; $\frac{A'_3}{A_3} = \frac{3\,R^2\sqrt{3}}{4} \div 3\,R^2\sqrt{3} = 1 : 4$; $\frac{s'_4}{s_4} = \frac{R\sqrt{2}}{2\,R} = \frac{\sqrt{2}}{2}$; $\frac{s'_6}{s_6} = R \div \frac{2\,R\sqrt{3}}{3} = \frac{\sqrt{3}}{2}$; $\frac{s'_3}{s_3} = \frac{R\sqrt{3}}{2\,R\sqrt{3}} = \frac{1}{2}$.

Note the correspondence of the various ratios. The ratios of the sides of the inscribed and the circumscribed equilateral triangles, squares, and regular hexagons are respectively $\frac{1}{2}$ $\left(\text{or } \frac{\sqrt{1}}{2}\right)$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$. The ratios of the areas of the inscribed and the circumscribed equilateral triangles, squares, and regular hexagons are respectively 1 : 4, 2 : 4: and 3 : 4.

14. $s_6 : s_3 = \frac{2\,R\sqrt{3}}{3} : 2\,R\sqrt{3} = 1 : 3$; $A_3 : A'_4 = 3\,R^2\sqrt{3} : 2\,R^2 = 3\sqrt{3} : 2$.

15. $(s_3 - s'_6) = 2\,R\sqrt{3} - R = R\,(2\sqrt{3} - 1) = 2.464\,R$; $(s'_3 - s'_4 = R\sqrt{3} - R\sqrt{2} = R\,(\sqrt{3} - \sqrt{2}) = .318\,R$; $(s_3 - s'_6)$ is the greater.

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Exercises. Circles—Sectors—Segments

1. $C = 12\,\pi = 37\frac{5}{7}$; $C = 20\,\pi = 62\frac{6}{7}$.

2. $S = 64\,\pi = 201\frac{1}{7}$; $S = 18\,\pi = 56\frac{4}{7}$; $S = 5\,a^2\,\pi = 15\frac{5}{7}\,a^2$.

3. $C = 24\,\pi = 75\frac{3}{7}$; $C = 50\,\pi = 157\frac{1}{7}$.

4. $C = 2\,\pi R$; $m = 2\,\pi R$; $R = \frac{m}{2\,\pi} = \frac{7\,m}{44}$.

5. (*a*) $C = 12\,\pi$; $p = 12 + 2\,\pi = 18\frac{3}{7}$; (*b*) $p = 12 + 3\,\pi = 21\frac{3}{7}$; (*c*) $p = 12 + \frac{m}{360} \times 12\,\pi = 12 + \frac{m\,\pi}{30}$.

6. $l = 2 \times 12' + \frac{3}{2}\,\pi = 24 + 4\frac{5}{7} = 28\frac{5}{7}'$.

7. Area of the circle $= 36\,\pi$; $S = \frac{1}{6} \times 36\,\pi = 6\,\pi = 18\frac{6}{7}$; $S = \frac{1}{4} \times 36\,\pi = 9\,\pi = 28\frac{2}{7}$; $S = \frac{m}{360} \times 36\,\pi = \frac{m\,\pi}{10} = \frac{11\,m}{35}$.

8. Area of sector $= \frac{1}{6} \times 25\pi = \frac{25\pi}{6}$; area of equilateral triangle $= (\frac{5}{2})^2\sqrt{3} = \frac{25}{4}\sqrt{3}$; area of segment $= \frac{25\pi}{6} - \frac{25\sqrt{3}}{4} = 2.27$.

9. Area of a cross section $= 4\pi$; rate of flow (in inches) $= 125 \times 12$; volume (in cu. in. per minute) $= 4\pi \times 125 \times 12 = 6{,}000\pi$; volume (in gallons per minute) $= 6{,}000 \times \frac{22}{7} \times \frac{1}{231} = 81.6$; volume (in gallons per hour) $= 81.6 \times 60 = 4{,}896$.

10. $R = 7$; $C = 14\pi = 44$; $n = \frac{5{,}280 \times 12}{44} = 1{,}440$.

11. $x = \frac{1}{3} \times 8\pi = \frac{176}{21} = 8.4$; $y = \frac{1}{3} \times 5\pi = \frac{110}{21} = 5.2$.

12. Perimeter of a semicircular segment $= 2R + \pi R = 5\frac{1}{7}R$; $5\frac{1}{7}R = 41\frac{1}{7}$; $R = 8$; area $= \frac{1}{2} \times \pi R^2 = 32\pi = 100\frac{4}{7}$.

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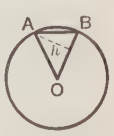
Exercises. Areas Determined by Trigonometric Functions

1. (*a*) $\sin S = \frac{h}{r}$; $\sin 70° = \frac{h}{8}$; $h = 8 \times .9397 = 7.5176$; area $= \frac{1}{2}ph = \frac{1}{2} \times 12 \times 7.5176 = 45.1$; (*b*) $\sin 50° = \frac{h}{10}$; $h = 10 \times .766 = 7.66$; area $= \frac{1}{2} \times 10 \times 7.66 = 38.3$; (*c*) $\sin 80° = \frac{h}{10}$; $h = 10 \times .9848 = 9.848$; area $= \frac{1}{2} \times 10 \times 9.848 = 49.24$.

2. (*a*) $\sin O = \frac{h}{OA}$; $\sin 50° = \frac{h}{10}$; $h = 10 \times .766 = 7.66$; let T = area of the triangle, S' = area of the sector, and S = area of the segment; $T = 38.3$; $S' = \frac{50}{360} \times 100\pi = 43.6$; $S = 5.3$; (*b*) $\sin 80° = \frac{h}{10}$; $h = 10 \times .9848 = 9.84$; $T = 49.2$; $S' = \frac{80}{360} \times 100\pi = 69.8$; $S = 20.6$; (*c*) $T = \frac{1}{2} \times 10 \times 10 = 50$; $S' = \frac{1}{4} \times 100\pi = 25\pi = 78\frac{4}{7}$; $S = 28\frac{4}{7}$.



3. Let PQ be a tangent to the circle; $\tan 40° = \frac{r}{PQ}$; PQ =

$\frac{r}{\tan 40^\circ} = 12 \div .839 = 14.3$; $T = \frac{1}{2} \times 12 \times 14.3 = 85.8$; quadrilateral $PQON = 2 \times 85.8 = 171.6$; sector $QON = \frac{100}{360} \times 144\,\pi =$ 125.7; the shaded portion = 45.9.

4. Let OA be the radius and OC the apothem of a regular inscribed pentagon; $OA = 6$; $\angle AOC =$ $36°$: $\sin 36° = \frac{AC}{6}$; $AC = 6 \times .5878 = 3.526$; $\cos$ $36° = \frac{OC}{6}$; $OC = 6 \times .809 = 4.8$; $S = \frac{1}{2}\,ap = \frac{1}{2} \times 4.8 \times 35.26$ $= 84.624$.

5. Use the figure of Ex. 4; $OA = 5$; $\angle AOC = 36°$; $\sin 36° = \frac{AC}{5}$; $AC = 5 \times .5878 = 2.939$; $\cos 36° = \frac{OC}{5}$; $OC = 5 \times .809 =$ 4.045; $\triangle OAB = 2.93 \times 4.04 = 11.837$; sector $OAB = \frac{1}{5}\,\pi 5^2 =$ $5\,\pi = 15.708$; segment $AB = 3.871$.

6. Use the figure of Ex. 4; let OC be the radius and the apothem of a regular circumscribed decagon; $OC = 6$; $\angle AOC = 18°$; tan $18° = \frac{AC}{OC}$; $AC = 6 \times .3249$; $AC = 1.9494$; $p = 20 \times 1.9494 =$ 38.988; $S = \frac{1}{2}\,ap = \frac{1}{2} \times 6 \times 38.988 = 116.964$.

7. Let AB be a side of a regular inscribed pentagon and CD be a side of an inscribed equilateral triangle; draw $OR \perp AB$ and CD; $OD = 6$; $\angle D = 30°$; OP $= 3$; $PD = 3\sqrt{3}$; $\triangle OCD = 9\sqrt{3} = 15.588$; sector $OCD = \frac{1}{3} \times 6^2\,\pi = 12\,\pi = 37.714$; segment $CD =$ 22.126; $OB = 6$; $\angle BOR = 36°$; $\angle OBR = 54°$; $\sin 54° = \frac{OR}{6}$; $OR = 6 \times .8090 = 4.854$; $\sin 36° = \frac{RB}{6}$; $RB = 6 \times .5878 =$ 3.5268; $\triangle OAB = 17.12$; sector $OAB = \frac{1}{5} \times 6^2\,\pi = 22.628$; segment $AB = 5.508$; $ABDC = 22.126 - 5.508 = 16.618$.

Optional Solution: A different method uses: $ABDC = 2$ (Sector $DOB) + \triangle OAB - \triangle OCD$; $\angle DOP = \frac{1}{2} \times \frac{1}{3} \times 360° = 60°$; $\angle BOR = \frac{1}{2} \times \frac{1}{5} \times 360° = 36°$; $\angle DOB = 24° = \frac{1}{15}$ of $360°$;

Sector $DOB = \frac{1}{15} \times$ area of the circle; $\triangle OAB = OR \times RB$; $\triangle OCD = OP \times PD$

8. Draw radius OC; $\angle B = 40°$; $\sin 40° = \frac{AC}{20}$; $AC = 20 \times .6428 = 12.856$; $\cos 40° = \frac{BC}{20}$; $BC = 20 \times .7660 = 15.32$; $\triangle ABC = \frac{1}{2} \times 12.856 \times 15.32 = 98.945$; $\triangle OBC = \triangle OAC = \frac{1}{2} \triangle ABC = 49.472$; $\angle BOC = 100°$; $\angle AOC = 80°$; sector $OBC = \frac{100}{360} \times 10^2 \pi = 87.26$; segment II $= 37.788$; sector $OAC = \frac{80}{360} \times 10^2 \pi = 69.81$; segment I $= 20.338$.

9. Let the common chord AB intersect OO' at C; $\angle COA = 40°$; $OA = 12$; $\angle CAO = 50°$; $\sin 40° = \frac{AC}{12}$; $AC = 12 \times .6428 = 7.71$; $\cos 40° = \frac{OC}{12}$; $OC = 12 \times .7660 = 9.19$; $\sin \angle CO'A = \frac{7.71}{8} = .96$; $\angle CO'A = 74°+$; $\angle O' = 148°$; $\cos 74° = \frac{O'C}{8}$; $O'C = 8 \times .2756 = 2.20$; $OO' = 9.19 + 2.20 = 11.39$; $OAO'B = 11.39 \times 7.71 = 87.86$; sector $OAB = \frac{80}{360} \times 12^2 \pi = 32\pi$; sector $O'AB = \frac{148}{360} \times 8^2 \pi = 26.3\pi$; the shaded area = sector OAB + sector $O'AB - OAO'B = 58.3\pi - 87.86 = 183.24 - 87.86 = 95.38$.

10. Use the figure of Ex. 1; let PH be the altitude; $\triangle PRS = \frac{1}{2} p \times PH$ (§ 242); $\sin S = \frac{PH}{r}$ (§ 330); $PH = r \sin S$ (*Prin.* 7); $\triangle PRS = \frac{1}{2} pr \sin S$ (Sub.).

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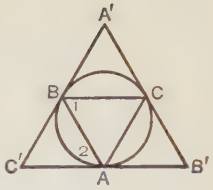
Supplementary Exercises. Regular Polygons

1. $BC = 4\sqrt{3}$; $BH = 2\sqrt{3}$; $\angle O = 60°$; $OH = 2\sqrt{3} \div \sqrt{3} = 2$; $\angle OBH = 30°$; $OB = 2 \times 2 = 4$.

2. Let CB and CA be respectively the sides of the hexagon and the triangle; $OA = AB = BC = OC$ (*Prin.* 33, § 346); $OD = DB = \frac{1}{2} OB$ (§ 171); $OD = \frac{1}{2} CB$ (Sub.).

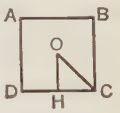
3. $AB = BC = CA$ (Given); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CA}$ (§ 196); $\angle 1 = \angle 2$ (§ 228); $BC \parallel C'A$ (§ 116); in like manner $CA \parallel C'B$; $BCAC'$ is a $\square$ (§ 138); $BC = C'A$ (§ 147); in like manner $BC = AB'$; $2\,BC = C'B'$ (*Prin.* 5).



4. See Ex. 5 (*a*), p. 277.

5. See Ex. 5 (*c*), p. 277.

6. See Ex. 31, p. 201. The formula of § 349 is used in finding the area of a regular polygon and does not apply in this case.

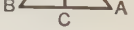


7. $AC = 6\sqrt{2}$; $OC = 3\sqrt{2}$; $OH = HC = 3$; $p = 24$; $S = \frac{1}{2}\,ap = \frac{1}{2} \times 3 \times 24 = 36$.

8. The apothem of a regular hexagon is the altitude of an equilateral triangle whose sides are equal to the sides of the hexagon; $h = \frac{s\sqrt{3}}{2}$; $\frac{a}{2}\sqrt{3} = \frac{s\sqrt{3}}{2}$; $a = s$; $p = 6\,s = 6\,a$; $S = \frac{1}{2}\,ap = \frac{1}{2} \times \frac{s\sqrt{3}}{2} \times 6\,s = \frac{3\,s^2\sqrt{3}}{2}$.

9. $\angle O = \frac{360°}{n}$; $\angle O = 120°;\ 90°;\ 72°;\ 60°;\ 45°;\ 36°$.

10. Let AB be a side of a regular polygon whose center is O; $AC = CB$ (Given); $OA = OB$ (*Prin.* 33); OC is the $\perp$ bisector of AB (§ 171); if an apothem were drawn from O to AB it would pass through C (§ 348); therefore, OC is an apothem (*Prin.* 13).



11. The regular hexagram is composed of twelve congruent equilateral triangles or $12 \times 5^2\sqrt{3} = 300\sqrt{3}$. The Facts required are §§ 71, 106, 108, 121, 158, 160, 166, 242, 321, 346, 348, and *Prins.* 3, 4, 7, 15, and 29.

Note: The better pupils might prove this exercise for an irregular hexagram as an extra credit example.

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Supplementary Exercises. Circles—Sectors—Segments

12. Use the figure for Ex. 1; $p = 30$; $s = 10$; $BD = 5$; $OD = \frac{5}{3}\sqrt{3}$; $OB = r = \frac{10}{3}\sqrt{3}$; $S = \pi r^2 = \frac{100}{3}\pi$.

13. $s = 12$; $\triangle ABC = \left(\frac{s}{2}\right)^2\sqrt{3} = 36\sqrt{3}$; the three 60° sectors equal a semicircle whose radius is 6; the semicircle $= \frac{1}{2}\pi 6^2 = 18\pi$; the shaded area $= 36\sqrt{3} - 18\pi = 18(2\sqrt{3} - \pi) = 18(3.464 - 3.141) = 18(.323) = 5.814$.

14. Area of the sector $= \frac{1}{4}\pi r^2$; area of the triangle $= \frac{1}{2}r^2$; area of a segment $= \frac{1}{4}\pi r^2 - \frac{1}{2}r^2$; area of the lune $= \frac{1}{2}\pi r^2 - r^2 = r^2\left(\frac{\pi}{2} - 1\right)$.

15. Wt $= 1120\,D^2 = 1120 \times 9 = 10{,}080$ lb. $= 5+$ tons.

16. Let R be the radius of the larger circle and d and r the diameter and the radius respectively of the smaller circle; $R = d = 2r$; $S = \pi R^2 = \pi(2r)^2 = 4\pi r^2$; $S' = \pi r^2$; $S : S' = 4\pi r^2 : \pi r^2 = 4 : 1$.

17. Let A and B be the areas of two sectors whose central angles are $a°$ and $b°$ respectively with r the radius of the circle; $A = \frac{a°}{360°}\pi r^2$ and $B = \frac{b°}{360°}\pi r^2$; $A : B = \frac{a}{360}\pi r^2 : \frac{b}{360}\pi r^2 = a : b$; $A : B = 2 : 3$; $a : b = 2 : 3$.

18. Let CD be a side of an inscribed equilateral triangle; $OD = R$; $\angle COD = 120°$; the area of sector $OCD = \frac{1}{3}\pi R^2$; $OP = \frac{1}{2}R$; $PD = \frac{1}{2}R\sqrt{3}$; $\triangle OCD = \frac{1}{4}R^2\sqrt{3}$; segment $CD = \frac{1}{3}\pi R^2 - \frac{1}{4}R^2\sqrt{3}$

$= R^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = R^2\left(\frac{4\pi}{12} - \frac{3\sqrt{3}}{12}\right) = \frac{R^2}{12}(4\pi - 3\sqrt{3})$.

19. Let AB be a side of a regular hexagon inscribed in a circle with center O; $\angle AOB = 60°$; sector $OAB = \frac{1}{6}\pi R^2$; $\triangle OAB = \frac{R^2\sqrt{3}}{4}$; segment $AB = \frac{\pi R^2}{6} - \frac{R^2\sqrt{3}}{4} = R^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = R^2\left(\frac{2\pi}{12} - \frac{3\sqrt{3}}{12}\right) = \frac{R^2}{12}(2\pi - 3\sqrt{3})$.

20. The total figure equals the equilateral triangle plus six 60° segments; $r = 10$; $a = 5$; $\frac{s}{2} = 5\sqrt{3}$; $s = 10\sqrt{3}$; $S = \left(\frac{10\sqrt{3}}{2}\right)^2\sqrt{3} = 75\sqrt{3}$; each 60° sector (whose radius is $\frac{s}{2} = 5\sqrt{3}$) $= \frac{1}{6}\pi(5\sqrt{3})^2 = \frac{75}{6}\pi = \frac{25}{2}\pi$; the triangle of each sector $= \left(\frac{5\sqrt{3}}{2}\right)^2\sqrt{3} = \frac{75}{4}\sqrt{3}$; each 60° segment $= \frac{25\pi}{2} - \frac{75\sqrt{3}}{4} = \frac{50\pi}{4} - \frac{75\sqrt{3}}{4} = \frac{25}{4}(2\pi - 3\sqrt{3})$; the six segments $= \frac{75}{2}(2\pi - 3\sqrt{3})$; the total area $= 75\sqrt{3} + \frac{75}{2}(2\pi - 3\sqrt{3}) = 75(\sqrt{3} + \pi - \frac{3}{2}\sqrt{3}) = 75(\pi - \frac{\sqrt{3}}{2})$.

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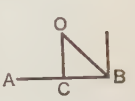
Miscellaneous Exercises

1. A central angle of a regular polygon of n sides $= \frac{360°}{n}$ (§ 345); an exterior angle of a regular polygon of n sides $= \frac{360°}{n}$ (§ 167); a central angle = an exterior angle (Sub.).

2. Let $AB = s$ and $OC = a$; $CB = \frac{1}{2}s$ (§ 348); $OC = a = \frac{1}{2}s$ (Given); $CB = OC = \frac{1}{2}s$ (Sub.); $\angle BOC = \angle OBC = 45°$; $\angle B = 90°$ (§ 348); $90° = \left(\frac{n-2}{n}\right)180°$; $1 = \frac{2(n-2)}{n}$; $n = 4$; the regular polygon is a square.



3. $K = \pi R^2$; $H = 2R^2\sqrt{3}$ and $h = \frac{3R^2\sqrt{3}}{2}$ (Exs. 10 and 9, p. 287); $\frac{H+h}{2} = \frac{4R^2\sqrt{3} + 3R^2\sqrt{3}}{4}$; $K : \frac{H+h}{2} = \pi R^2 : \frac{4R^2\sqrt{3} + 3R^2\sqrt{3}}{4} = \pi : \frac{7}{4}\sqrt{3}$.

4. Let R and r be radii of the larger and the smaller circles respectively and let AB be a chord of the larger circle tangent to the smaller circle at C; let S and S' be the areas of the larger and the smaller circles respectively; $S = \pi R^2$ and $S' = \pi r^2$ (§ 363); the

area of the ring $= S - S' = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$ (Sub.); $R^2 = r^2 + (AC)^2$ (§ 321); $AC = \frac{1}{2} AB$ (§ 200); $R^2 - r^2 = (AC)^2$ (*Prin.* 6); the area of the ring $= \pi (R^2 - r^2) = \pi (AC)^2$ (Sub.).

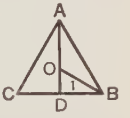
5. See Ex. 4 and Ex. 10, p. 287.

6. The total area equals the area of the square plus four semicircles or two circles whose radii are $\frac{s}{2}$; the total area $= s^2 + 2\pi\left(\frac{s}{2}\right)^2 = s^2 + \frac{\pi s^2}{2} = s^2\left(1 + \frac{\pi}{2}\right)$.

7. $S = \pi R^2 = 12\pi$; $R^2 = 12$; $R = 2\sqrt{3}$; $s = R = 2\sqrt{3}$; $h = \frac{s}{2}\sqrt{3}$; $h = \frac{2\sqrt{3}}{2} \cdot \sqrt{3} = 3$.

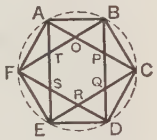
8. $S = \pi r^2 = 32\pi$; $r = 4\sqrt{2}$; diameter $= 8\sqrt{2}$; area of $\triangle = \frac{1}{2} \times 4\sqrt{2} \times 8\sqrt{2} = 32$.

9. Let ABC be an inscribed equilateral trangle with $h = AD$ and $r = OB = OA$; $\angle 1 = 30°$ (§ 348); $OD = \frac{1}{2} OB$ (§ 158); $h = AD = OA + OD = OB + \frac{1}{2} OB = r + \frac{1}{2} r = \frac{3}{2} r = \frac{3}{4} d$ (Sub.).



10. Use the figure of Ex. 9; let $OD = r$; $\angle 1 = 30°$ (§ 348); $OB = 2\,OD$ (§ 158); $OB = 2r$ (Sub.); $OA = OB$ (*Prin.* 33); $h = AD = OD + OA = r + 2r = 3r$ (Sub.).

11. Let $ABCDEF$ be a regular hexagon whose diagonals form hexagon $OPQRST$; let a circle be circumscribed about the regular polygon (§ 338); $AB = BC = CD =$ etc. (§ 160); $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} =$ etc. $= \frac{1}{6} \times 360° = 60°$ (§§ 196, 191); $\angle EAC =$ numerically $\frac{1}{2}\overset{\frown}{EC} = \frac{1}{2} \times 120° = 60°$ (§ 224 and Sub.); $\angle ATB =$ numerically $\frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{FE}) = \frac{1}{2}(120°) = 60°$ (§ 229 and Sub.); in like manner $\angle AOT = 60°$; $AT = AO = OT$ (§ 121); in like manner $\angle EAB =$ numerically $\frac{1}{2}\overset{\frown}{BE} = \frac{1}{2}(180°) = 90°$ (§ 224 and Sub.); $AT = \frac{AB}{\sqrt{3}} = \frac{s}{\sqrt{3}}$ (§ 329); $OT = \frac{s}{\sqrt{3}}$ (Sub.); in like manner $OT = OP = PQ =$ etc. $= \frac{s}{\sqrt{3}}$; $\angle TOP = \angle ATO + \angle TAO = 120°$ (§ 110 and Sub.); in like manner $\angle TOP = \angle OPQ = \angle PQR =$ etc. $= 120°$; $OPQRST$ is a regular polygon (§ 160).



12.	s	r	a	S
(a) square	5	$2\frac{1}{2}\sqrt{2}$	$2\frac{1}{2}$	25
(b) regular hexagon	6	6	$3\sqrt{3}$	$54\sqrt{3}$
(c) regular triangle	8	$\frac{2}{3}\sqrt{3}$	$\frac{4}{3}\sqrt{3}$	$16\sqrt{3}$
(d) square:	s	$\frac{1}{2}s\sqrt{2}$	$\frac{1}{2}s$	s^2
(e) regular hexagon	s	s	$\frac{s}{2}\sqrt{3}$	$\frac{3s^2\sqrt{3}}{2}$
(f) regular triangle	s	$\frac{s}{3}\sqrt{3}$	$\frac{s}{6}\sqrt{3}$	$\frac{s^2\sqrt{3}}{4}$

13. $S = (\frac{8}{9}D)^2 = (\frac{16}{9}R)^2 = \frac{256}{81}R^2 = \pi R^2$; $\pi = \frac{256}{81} = 3.1604$.

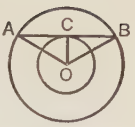
14. Since the area of the 40-cent pie (36π sq. in.) is more than twice the area (16π sq. in.) of the 20-cent pie, the larger pie is the better bargain.

15. (a) Let S equal the rate in miles an hour with the original tires and S' equal the rate in miles an hour with the oversize tires for any given speed of the motor. Then $S : S' = 30 : 31$; $30S' = 31S$; $S' = \frac{31}{30}S$. In the given case, $S' = \frac{31}{30} \times 30 = 31$. ($b$) If the tire diameters are A and B inches respectively, $S : S' = A : B$; $B \times S = A \times S'$; $S = \frac{A}{B}S'$. Since $\frac{A}{B} \times$ a number equals that number minus $\frac{B-A}{B} \times$ the number (or $\frac{A}{B} \times N = N - \left(\frac{B-A}{B}\right)N$); therefore $S = \frac{B-A}{B}$ of S' slower than S' or $\frac{100(B-A)}{B}\%$ slower.

16. $A_3 = ks^2 = \frac{s^2}{4}\sqrt{3}$; $k = \frac{1}{4}\sqrt{3}$; $A_4 = ks^2 = s^2$; $k = 1$; $A_6 = ks^2 = 3\frac{s^2\sqrt{3}}{2}$; $k = \frac{3}{2}\sqrt{3}$; $A_8 = ks^2 = 2s^2(\sqrt{2}+1)$; $k = 2(\sqrt{2}+1)$.

Note: The area of a regular octagon whose side is s equals the area of a square whose side is $s(1+\sqrt{2})$ minus the area of four isosceles right triangles whose equal legs are $\frac{s}{2}\sqrt{2}$ or $[s(1+\sqrt{2})^2] - 4\left[\frac{1}{2}\cdot\left(\frac{s}{2}\sqrt{2}\right)^2\right] = s^2(1+2+2\sqrt{2}) - s^2 = 2s^2(\sqrt{2}+1)$.

17. $\overset{\frown}{AB} = 120°$; $\angle AOB = 120°$; $OC \perp AB$; $\angle COB = 60°$; $\angle OBC = 30°$; $OC = \frac{1}{2}\,OB$; $S : S' = (OB)^2 : (OC)^2 = (OB)^2 : (\frac{1}{2}\,OB)^2 = 4 : 1$; $C : C' = OB : OC = OB : \frac{1}{2}\,OB = 2 : 1$.



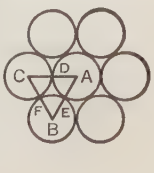
18. $s_3 = 8$; $A_3 = 16\sqrt{3} = 27.712$; $s_4 = 6$; $A_4 = 36$; $s_6 = 4$; $A_6 = 24\sqrt{3} = 41.568$; $C = 24 = 2\,\pi R$; $R = \dfrac{12}{\pi}$; $A = \pi R^2 = \dfrac{144}{\pi} = 45.81$.

Note: The areas increase as the number of sides increases while the perimeter remains constant. It can be shown in geometry that of all plane figures with equal perimeters the circle has the maximum area.

19. Let s represent a side, a an apothem, and r a radius of each regular polygon; let S and S' represent the areas of the circles circumscribed about and inscribed in each regular polygon; (*a*) in the equilateral triangle, $r = 2\,a$; $S = \pi r^2 = \pi\,(2\,a)^2 = 4\,\pi a^2$; $S' = \pi a^2$; $S : S' = 4\,\pi a^2 : \pi a^2 = 4 : 1$; (*b*) in the square, $r = a\sqrt{2}$; $S = \pi r^2 = \pi\,(a\sqrt{2})^2 = 2\,\pi a^2$; $S' = \pi a^2$; $S : S' = 2\,\pi a^2 : \pi a^2 = 2 : 1$; (*c*) in the regular hexagon, $a = \frac{1}{2}\,r\sqrt{3}$; $S = \pi r^2$; $S' = \pi a^2 = \pi\,(\frac{1}{2}\,r\sqrt{3})^2 = \frac{3}{4}\,\pi r^2$; $S : S' = \pi r^2 : \frac{3}{4}\,\pi r^2 = 4 : 3$.

Note: The ratios of the areas of the inscribed to the circumscribed circles for the equilateral triangle, square, and regular hexagon are respectively 1 : 4, 2 : 4, and 3 : 4. Compare these results with the ratios in Ex. 13, p. 287. Observe that the ratio of the areas of an inscribed and a circumscribed equilateral triangle and the ratio of the areas of circles inscribed in and circumscribed about an equilateral triangle is in each case 1 : 4. A like correspondence exists also for squares and regular hexagons.

20. The area of the inner figure equals the area of a hexagon minus the areas of six sectors whose central angles are each 120°. Let $CF = r$; $CB = 2\,r$; $AF = r\sqrt{3}$; the hexagon $= \frac{1}{2} \times 6 \times 2\,r \times r\sqrt{3} = 6\,r^2\sqrt{3}$; the six sectors $= 6 \times \frac{120}{360} \times \pi r^2 = 2\,\pi r^2$; the area of the inner figure $= 6\,r^2\sqrt{3} - 2\,\pi r^2 = 2\,r^2(3\sqrt{3} - \pi)$.



CHAPTER SIX

This chapter deals with three distinct units of work, namely, Inequalities, Constructions, and Loci. There are a number of good reasons for giving these parts of geometry a separate and unified treatment. In place of the usual mixture throughout the text, which is both time-wasting and difficult, there is a straight line drive which gains in power and momentum as the course progresses, not only in the first five chapters, but also in Chapter 6. It will be observed that only one proposition of Inequalities (Proposition 1, § 376) is required as a standard book theorem by the College Entrance Examination Board and that the National Committee requires none. The New York Regent's Syllabus (1927) lists only two (§§ 378, 380) as required theorems. The other theorems under Inequalities are subsidiary; they may be treated as original exercises and all that should be required in some instances is that the statements of the theorems should be memorized. Since the majority of the theorems of geometry have been studied in the first five chapters, it will be found that some of them may be used to advantage in the proofs of Chapter 6. In this connection, the Pythagorean Theorem lends itself for simpler proofs and in one instance (Proposition 8, § 386) for a more rigorous proof than is found in many other texts. Possibly the most important matter which should receive attention while studying Inequalities is the subject of Indirect Proofs. Converse theorems and Inequalities are frequently proved more easily by indirect methods than by direct and because of the importance of such fundamental methods of reasoning, they may well be emphasized at this time. For those who wish to take Inequalities throughout the chapters, the Optional Sequence on p. viii tells how this may be done.

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Exercises. Inequalities in Polygons

1. If $\angle A$ is obtuse, $\angle A > 90°$ (§ 31); $\angle A + \angle B + \angle C = 180°$ (§ 106); $\angle B + \angle C < 90°$ (*Prin.* 47); $\angle B + \angle C > \angle B$ or $\angle C$ (*Prin.* 48); $\angle A > 90° > \angle B + \angle C > \angle B$ or $\angle C$ (*Prin.* 49).

2. $AB = AC$ (Given); $\angle A$ is obtuse (Given); $\angle B = \angle C$ (§ 79); $\angle A > \angle B$ or $\angle C$ (Ex. 1); $BC > AB$ or AC (§ 380).

3. Let $ABCDEF$ and ACE be the hexagon and the triangle; $AB + BC > AC$, $CD + DE > CE$, and $EF + FA > EA$ (*Prin.* 22); $AB + BC + CD + DE + EF + FA > AC + CE + EA$ (*Prin.* 45).

4. Let AD be the altitude of $\triangle ABC$ (Given); $AD < AB$ and $AD < AC$ (*Prin.* 23); $2\,AD < AB + AC$ (*Prin.* 45); $AD < \frac{1}{2}(AB + AC)$ (*Prin.* 44).

5. $AB = AC$ (Given); (*a*) $AB + AD > BD$ (*Prin.* 22); $AC + AD$ or $CD > BD$ (Sub.); (*b*) $\angle C = \angle ABC$ (§ 80); $\angle ABC > \angle S$ (§ 373); $\angle C > \angle S$ (*Prin.* 43); $AS > AC$ (§ 380); (*c*) $\angle ARC > \angle ABC$ (§ 373); $\angle ARC > \angle C$ (*Prin.* 43); $AC > AR$ (§ 380).

6. $\angle A = 50°$, $\angle B = 60°$, and $\angle C = 70°$ (Given); $AB > AC$, $AC > BC$ (§ 380); $AB > AC > BC$ (*Prin.* 49).

7. $\angle B > \angle C > \angle A$ (Given); $AC > AB$, $AB > BC$ (§ 380); $AC > AB > BC$ (*Prin.* 49).

8. $CD = DB$ (Given); $\angle ADB < 90°$ (Given); $\angle CDB = 180°$ (§ 28); $\angle ADC > 90°$ (*Prin.* 47); $AD = AD$ (Iden.); $AC > AB$ (§ 381).

9. $CD = DB$ (Given); $AD = AD$ (Iden.); $AC > AB$ (Given); $\angle ADC > \angle ADB$ (§ 382); $\angle ADC + \angle ADB = 180°$ (*Prin.* 29); $\angle ADB = 180° - \angle ADC$ (*Prin.* 6); $\angle ADC > 180° - \angle ADC$ (Sub.); $2\,\angle ADC > 180°$ (*Prin.* 5); $\angle ADC > 90°$ (*Prin.* 8); $\angle ADC$ is obtuse (§ 31).

10. Use the figure of Ex. 17, p. 307, and let the diagonals intersect at O; $OA + OB > AB$ and $OC + OD > CD$ (*Prin.* 22); $AC + BD > AB + CD$ (*Prin.* 45).

11. $ABCD$ is a $\square$ (Given); $\angle A > 90°$ (Given); $\angle A + \angle D = 180°$ (§ 145); $\angle D < 90°$ (*Prin.* 47); $\angle A > 90° > \angle D$ (*Prin.* 49); $\angle D > \angle ADB$ (*Prin.* 48); $\angle A > \angle D > \angle ADB$ (*Prin.* 49); $DB > AB$ (§ 380) and likewise $DB > AD$.

12. Let $ABCD$ and $A'B'C'D'$ be mutually equilateral parallelograms; $\angle A' < \angle A$ (Given); $B'D' < BD$ (§ 381).

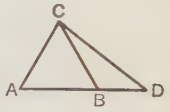
13. Use the figure of Ex. 17 and let the diagonals intersect at O; $AC < AB + BC$ and $AC < CD + DA$ (*Prin.* 22); $2\,AC <$ per-

imeter p (*Prin.* 45); $AC < \frac{1}{2}\,p$ (*Prin.* 44) and likewise $BD < \frac{1}{2}\,p$; $AC + BD < p$ (*Prin.* 45). $OA + OB > AB$, $OB + OC > BC$, $OC + OD > CD$, and $OA + OD > AD$ (*Prin.* 22); $2\,(AC + BD) > p$ (*Prin.* 45); $AC + BD > \frac{1}{2}\,p$ (*Prin.* 44).

14. $AB + BD > AD$ (*Prin.* 22); $CD = CD$ (Iden.); $AB + BC > AD + CD$ (*Prin.* 44).

15. Let x, y, and z be the altitudes from A, B, and C respectively; $x < \frac{1}{2}\,(AB + AC)$, $y < \frac{1}{2}\,(AB + BC)$, and $z < \frac{1}{2}\,(AC + BC)$ (Ex. 4, p. 306); $x + y + z < p$ (*Prin.* 45).

16. $AB = BC = AC$ (Given); $\angle A = \angle ABC = \angle BCA$ (§ 81); $\angle ABC > \angle D$ (§ 373); $\angle ACD > \angle BCA$ (*Prin.* 48); $\angle ACD > \angle BCA = \angle A = \angle ABC > \angle D$ (*Prins.* 43 and 49); $AD > CD$, $CD > AC$ (§ 380); $AD > CD > AC$ (*Prin.* 49).

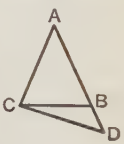


17. (*a*) $BC = AD$ (Given); $DC = DC$ (Iden.); $\angle D > \angle C$ (Given); $AC > BD$ (§ 381); (*b*) in $\triangle ABC$ and $\triangle ABD$, $BC = AD$ (Given); $AB = AB$ (Iden.); $AC > BD$ (Part *a*); $\angle B > \angle A$ (§ 382).

18. Use the figure of Ex. 11 with diagonals BD and AC of $\square ABCD$ (Given); $BD > AC$ (Given); $AD = BC$ (§ 147); $AB = AB$ (Iden.); $\angle A > AB$ (§ 382); $\angle A$ is obtuse and $ABCD$ is not a rectangle (Ex. 9, p. 307).

19. $ABCD$ is a rectangle (Given); $(PD)^2 = (PC)^2 + (DC)^2$ and $(AC)^2 = (AD)^2 + (DC)^2$ (§ 321); $PC > BC$ (*Prin.* 48); $BC = AD$ (§ 147); $PC > AD$ (Sub.); $(PC)^2 > (AD)^2$ (*Prin.* 46); $(DC)^2 = (DC)^2$ (Iden.); $(PC)^2 + (DC)^2 > (AD)^2 + (DC)^2$ (*Prin.* 44); $(PD)^2 > (AC)^2$ (*Prin.* 43); $PD > AC$ (*Prin.* 46).

20. $AB = AC$ (Given); $\angle ACB = \angle ABC$ (§ 80); $\angle ABC > \angle D$ (§ 373); $\angle ACD > \angle ACB$ (*Prin.* 48); $\angle ACD > \angle ABC$ (*Prin.* 43); $\angle ACD > \angle D$ (*Prin.* 49).



21. $AC < BC$ (Given); $\angle ABC < \angle BAC$ (§ 378); $AB = AB$ (Iden.); $BE = AD$ (Given); $AE < BD$ (§ 381).

22. Let O be a point within $\triangle ABC$; $OA + OB > AB$, $OB + OC > BC$, and $OC + OA > AC$ (*Prin.* 22); $2\,(OA + OB + OC) > p$ (*Prin.* 45); $OA + OB + OC > \frac{1}{2}\,p$ (*Prin.* 44).

23. Produce CP to intersect AB at D (*Prin.* 11); $AC + AD > PC + PD$ and $PD + DB > PB$ (*Prin.* 22); $AC + AB + PD > PC + PB + PD$ (*Prin.* 45); $PD = PD$ (Iden.); $AC + AB > PC + PB$ (*Prin.* 44).

24. $PA \perp AC$ (Given); $AC > AB$ (Given); $(PC)^2 = (PA)^2 + (AC)^2$ and $(PB)^2 = (PA)^2 + (AB)^2$ (§ 321); $(AC)^2 > (AB)^2$ (*Prin.* 46); $(PA)^2 = (PA)^2$ (Iden.); $(AC)^2 + (PA)^2 > (AB)^2 + (PA)^2$ (*Prin.* 44); $(PC)^2 > (PB)^2$ (*Prin.* 43); $PC > PB$ (*Prin.* 46).

25. $PA \perp AC$ (Given); $PC > PB$ (Given); $(PC)^2 = (PA)^2 + (AC)^2$ and $(PB)^2 = (PA)^2 + (AB)^2$ (§ 321); $(PC)^2 > (PB)^2$ (*Prin.* 46); $(PA)^2 + (AC)^2 > (PA)^2 + (AB)^2$ (*Prin.* 43); $(PA)^2 = (PA)^2$ (Iden.); $(AC)^2 > (AB)^2$ (*Prin.* 44); $AC > AB$ (*Prin.* 46).

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Exercises. Indirect Proofs

1. $XP = PY = \frac{1}{2}\,XY$ (Given); assume that P' is also a mid-point; then $XP' = P'Y = \frac{1}{2}\,XY$; $XP = XP'$ (Sub.); P and P' coincide (§ 23) and there can be only one mid-point of a line.

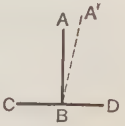
2. The proof corresponds to that of Ex. 1.

3. Let $\angle C = 90°$ in $\triangle ACB$ (Given); assume that $\angle A = 90°$ also; then BC and BA are both $\perp CA$, which is impossible (*Prin.* 15); therefore, the triangle may have only one right angle.

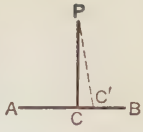
4. $\angle 1 = \angle 2$ and $\angle ROD = \angle ROB = \frac{1}{2}\,\angle BOD$ (Given); assume that POR' is a st. line, then $\angle R'OD = \angle R'OB = \frac{1}{2}\,\angle BOD$ (Ex. 1, p. 38); $\angle ROD = \angle R'OD$ (Sub.); OR' coincides with OR (§ 23) and since POR' is a straight line, POR is also a straight line.

5. Draw DE (*Prin.* 13); assume that BD and CE bisect each other, then $BCDE$ is a $\square$ (§ 152*b*); and $CD \parallel BE$ (§ 138); but CD and BE meet at A, therefore, BD and CE do not bisect each other.

6. $AB \perp CD$ (Given); $\angle ABC =$ rt. $\angle$ (§ 27); assume that $A'B \perp CD$, then $\angle A'BC =$ rt. $\angle$ (§ 27); $\angle A'BC = \angle ABC$ (Sub.); and $A'B$ coincides with AB (§ 23); so there is only one perpendicular to a line at a given point.



7. $PC \perp AB$ (Given); assume that $PC' \perp AB$; $\angle PCB$ = rt. $\angle$ and $\angle PC'A$ = rt. $\angle$ (§ 27); $\angle PCB + \angle PC'A$ = st. $\angle$ (*Prin.* 5); but $\angle PCB + \angle PC'A + \angle P$ = st. $\angle$ (§ 106); therefore, there is only one perpendicular to a line at a given point.



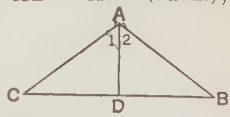
8. $\angle A > 90°$ (Given); $\angle A + \angle B + \angle C = 180°$ (§ 106); $\angle B + \angle C < 90°$ (*Prin.* 47); $\angle B$ or $\angle C < \angle B + \angle C < 90°$ (*Prins.* 48 and 49).

9. $\angle A$ is the vertex angle of an isosceles triangle with base angles, $\angle B$ and $\angle C$ (Given); $\angle B = \angle C$ (§ 80); assume that $\angle B > 90°$; then $\angle C > 90°$ (*Prin.* 43) and $\angle B + \angle C > 180°$ (*Prin.* 45); but $\angle A + \angle B + \angle C = 180°$ (§ 106); therefore, the assumption is contrary to fact and only the vertex angle of an isosceles triangle may be obtuse.

10. $AB = AC$ (Given); $\angle B = \angle ACB$ (§ 80); assume that $AC' = AC = AB$, then $\angle B = \angle AC'B$ (§ 80); $\angle AC'B = \angle ACB$ (Sub.); but $\angle AC'B > \angle ACB$ (§ 373); therefore, the assumption is false and only two equal lines can be drawn from a point to a line. (A similar proof may be given if C' falls on BC produced.)

11. $\angle C > \angle B$ (Given); assume $AB = AC$, then $\angle C = \angle B$ (§ 80), which is contrary to fact; assume $AB < AC$, then $\angle C < \angle B$ (§ 378), which is also contrary to fact; therefore, $AB > AC$ (*Prin.* 50).

12. Assume $CD > DB$ (or $CD < DB$); $AD = AD$ (Iden.); $AC = AB$ (Given); $\angle 1 > \angle 2$ (or $\angle 1 < \angle 2$) (§ 382); but $\angle 1 = \angle 2$ (Given); therefore, either assumption is false and the theorem is true.



13. $ax = b$ (Given); either $x = \frac{b}{a}$ or $x > \frac{b}{a}$ or $x < \frac{b}{a}$ (*Prin.* 50); if $x > \frac{b}{a}$, then $ax > b$ (*Prin.* 44), which is contrary to fact; if $x < \frac{b}{a}$, then $ax < b$ (*Prin.* 44), which is also contrary to fact; therefore, $x = \frac{b}{a}$ (*Prin.* 50).

14. $\angle A$ = rt. $\angle$ and $\angle B$ = rt. $\angle$ (Given); assume that $\angle A = \angle B \neq x°$; $\angle A$ = rt. $\angle \neq x°$ (Sub.); this is impossible since $\angle A$ = rt. $\angle$ (Given); therefore, the assumption is false and $\angle A = \angle B$.

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Exercises. Inequalities in Circles

1. $BD > AC$ (Given); $\overset{\frown}{BD} > \overset{\frown}{AC}$ (§ 385); $\overset{\frown}{BC} = \overset{\frown}{BC}$ (Iden.); $\overset{\frown}{CD} > \overset{\frown}{AB}$ (*Prin.* 44); $CD > AB$ (§ 385).

2. $CD > AB$ (Given); $\overset{\frown}{CD} > \overset{\frown}{AB}$ (§ 385); $\overset{\frown}{BC} = \overset{\frown}{BC}$ (Iden.); $\overset{\frown}{BD} > \overset{\frown}{AC}$ (*Prin.* 44); $BD > AC$ (§ 385).

3. $\angle B = 40°$ and $\angle C = 60°$ (Given); $\angle D + \angle B + \angle C = 180°$ (§ 106); $\angle D = 80°$ (Sub. and *Prin.* 6); $\angle D > \angle C > \angle B$ (Sub. since $80° > 60° > 40°$); $BC > BD > CD$ (§ 380); $\overset{\frown}{BC} > \overset{\frown}{BD} > \overset{\frown}{CD}$ (§ 385).

4. $OA = OC$ (Given); $\overset{\frown}{OA} = \overset{\frown}{OC}$ (§ 196); $\angle 1 > \angle 2$ (Given); $\angle 1$ = numerically $\frac{1}{2}\overset{\frown}{BC}$ and $\angle 2$ = numerically $\frac{1}{2}\overset{\frown}{AB}$ (§ 224); $\frac{1}{2}\overset{\frown}{BC} > \frac{1}{2}\overset{\frown}{AB}$ (Sub.); $\overset{\frown}{BC} > \overset{\frown}{AB}$ (*Prin.* 44); arc BCO > arc BAO (*Prin.* 44).

5. $\angle P > \angle R$ (Given); $RS > PS$ (§ 380); $\overset{\frown}{RS} > \overset{\frown}{PS}$ (§ 385).

6. $\overset{\frown}{RS} > \overset{\frown}{PS}$ (Given); $RS > PS$ (§ 385); $\angle P > \angle R$ (§ 378).

7. Draw OB (*Prin.* 13); $PO + OB > PB$ (*Prin.* 22); $OB = OA$ (*Prin.* 33); $PO + OA$ or $PA > PB$ (Sub.).

8. AC is a diameter (Given); arc ADC = arc ABC (*Prin.* 36); $\angle 1 > \angle 2$ (Given); $\angle 1$ = numerically $\frac{1}{2}\overset{\frown}{DC}$ and $\angle 2$ = numerically $\frac{1}{2}\overset{\frown}{CB}$ (§ 224); $\overset{\frown}{DC} > \overset{\frown}{CB}$ (Sub. and *Prin.* 44); $\overset{\frown}{AD} < \overset{\frown}{AB}$ (*Prin.* 47); $AD < AB$ (§ 385).

9. $\angle 1 > \angle 2$ (Given); $\angle 1$ = numerically $\frac{1}{2}\overset{\frown}{x}$ and $\angle 2$ = numerically $\frac{1}{2}\overset{\frown}{y}$ (§ 224); $\overset{\frown}{x} > \overset{\frown}{y}$ (Sub. and *Prin.* 44). The greater inscribed angle of a circle (or of equal circles) has the greater arc; conversely, the greater arc has the greater inscribed angle.

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Exercises. Inequalities in Circles

1. GH and KL are $\perp OS$ (Given); $OR > OP$ (*Prin.* 48); $KL < GH$ (§ 386).

2. Radius $OR \perp AB$ (Given); let CD be any other chord through P with $OH \perp CD$; $OP > OH$ (*Prin.* 23); $AB < CD$ (§ 386).

3. $OG \perp BA$ and $OH \perp AC$ (Given); rt. $\angle OGA$ = rt. $\angle OHA$ (*Prin.* 26); $\angle 1 > \angle 2$ (Given); $\angle OHG < \angle OGH$ (*Prin.* 47); $OG < OH$ (§380); $AB > AC$ (§386).

CONSTRUCTIONS

Note 1: Among the Principles are listed all those statements which are necessary (singly or in combination) to postulate the existence of the fundamental figures of geometry. When such figures are drawn freehand or with the aid of instruments, more or less crude representations are made. This has been the purpose of our drawing work in previous chapters. In the constructions of Chapter Six, we are to teach pupils how to make certain standard and more complicated figures and how to prove that such figures are correct when they are considered ideally.

Even though pupils may have had drawing work previous to the present course, it will be found advisable to take up rather carefully the fourteen Preliminary Constructions (pp. 315–317), since they not only lay a sound foundation, but also tend to take the guess work out of later problems.

Note 2: In the Manual and Key, due to the simple character of many of the problems, hints for their solutions are suggested; for those which are more complicated, fuller explanations are given. **The constructions made by pupils should always show all the necessary construction arcs and lines.** Such construction lines and marks are not shown in the figures given in the Manual and Key because they are not essential to the understanding of the solutions of the problems.

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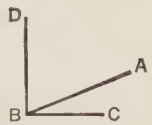
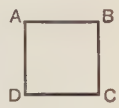
Construction Exercises. Perpendicular Lines

1. At the vertex of the given angle, $\angle ABC$, construct $BD \perp BC$ (§ 395, Part 1); $\angle ABD$ is the required complement.



2. Construct $\angle D$ = rt. $\angle$; measure DA and DC equal to the given line; describe arcs with A and C as centers and with radii equal to the given line, letting the arcs intersect at B; $ABCD$ is the required square.



3. Use § 395, Part II.

4. Construct a rt. angle; measure equal segments on its sides; join their extremities; the triangle is a 45-45-90 triangle.

5. Use § 395, Part I.

6. Construct a right triangle whose hypotenuse is double a leg.

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Construction Exercises. Triangles

1. Use § 399, Part I.

2. Use § 399, Part II.

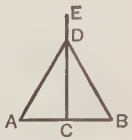
3. Use § 399, Part II.

4. Use § 400 with $\angle A$ = rt. $\angle$ and AC corresponding to the given leg.

5. Use § 399, Part I.

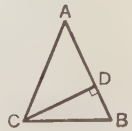
6. Use § 398.

7. Use § 399, Part II.



8. Draw base AB (Given); construct EC, the perpendicular bisector of AB (§ 393); on CE measure CD (Given); draw DA and DB.

9. With CD and CB (Given) use Ex. 4, p. 324, to find rt. $\triangle BDC$; at C construct $\angle ACB = \angle B$ (§ 396); produce BD to meet CA.



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Construction Exercises. Angles

1. Construct a rt. angle and bisect it.

2. Construct an equilateral triangle (Prelim. Cons. 11) and bisect an angle.

3. Add a 45° angle and a 30° angle.

4. Add a 90° angle and a 30° angle or find the supplement of a 60° angle.

5. Find the supplement of a 45° angle.

6. Bisect the given angle and then bisect its half.

7. Construct two equal adjacent angles each equal to the given angle (§ 396).

8. Construct a rt. angle ABC; within the rt. angle construct equilateral triangles whose sides are BA and BC respectively; the rt. angle is thus trisected.

9. Use §§ 396 and 401.

10. Bisect the supplement of the angle at the vertex.

11. Double the base angle and find the supplement of the doubled angle.

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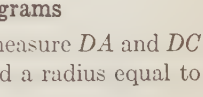
Construction Exercises. Parallelograms

1. Construct $\angle D$ equal to the given angle; measure DA and DC equal to the given sides; with A as a center and a radius equal to DC describe an arc; with C as a center and a radius equal to DA describe an arc intersecting the other arc at B; $ABCD$ is the required parallelogram.



2. Find a triangle whose sides equal DA, DC, and AC; complete the parallelogram as in Ex. 1.

3. Construct a 60° angle; complete the rhombus as in Ex. 1.

4. Bisect the perimeter and again bisect the half perimeter; construct a 90° angle; complete the square as in Ex. 1.

5. Let the diagonals of the rhombus be AC and BD; construct $x \perp y$ at O; on x and y measure $OA = OC = \frac{1}{2} AC$ and $OB = OD = \frac{1}{2} BD$; join A, B, C, and D; $ABCD$ is the required rhombus.

6. Find a right triangle whose hypotenuse equals a given line (AD) and one of whose legs equals a given line (AE) as in Ex. 4, p. 324; measure DC equal to the other given side; complete the parallelogram as in Ex. 1.

7. The diagonals of a square are equal and are the perpendicular bisectors of each other. The construction is similar to that of Ex. 5.

8. Let AC and BD be the given diagonals and AB the given side; find a triangle whose sides equal AB, $AO = \frac{1}{2} AC$, and $BO = \frac{1}{2} BD$; produce AO to C making $OC = OA = \frac{1}{2} AC$ and produce BO to D making $OD = OB = \frac{1}{2} BD$; join B, C, D, and A; $ABCD$ is the required parallelogram.

9. Let AC and BD be the given diagonals and $\angle 1$ the given angle; find a triangle in which two sides equal $OA = \frac{1}{2} AC$ and $OB = \frac{1}{2} BD$ with $\angle 1$ their included angle; complete the parallelogram as in Ex. 8.

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Construction Exercises

1. Find a right triangle ADC in which the hypotenuse equals the given side AC and a leg equals the given altitude AD (Ex. 4, p. 324); produce CD making $DB = CD$; draw AB; ABC is the required isosceles triangle.

2. Construct $x \perp y$ at D; measure DA on x; at A draw $\angle CAD = \angle DAB = \frac{1}{2}$ the given angle; $\triangle ABC$ is the required isosceles triangle.

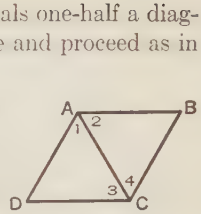
3. Since the median to the base of an isosceles triangle is also the altitude upon the base, this exercise corresponds to Ex. 8, p. 325.

4. Since the median to the base of an isosceles triangle is an altitude, find rt. $\triangle ADC$ with AD equal to the given median and $\angle C$ equal to the given angle; produce CD making $DB = CD$; draw AB; $\triangle ABC$ is the required isosceles triangle.

5. The radius of the inscribed circle equals half a side of the square (Ex. 2, p. 287); find a side of the square and use the method suggested for Ex. 4, p. 326.

6. The radius of the circumscribed circle equals one-half a diagonal of a square; find the diagonal of the square and proceed as in Ex. 7, p. 327.

7. In a rhombus $\angle A = \angle C$ (§ 145) and $\angle 1 = \angle 2 = \angle 3 = \angle 4 = \frac{1}{2}\angle A$ (Ex. 8, p. 91); find isosceles triangles ACD and ACB on opposite sides of AC (§ 399, I); $ABCD$ is the required rhombus.



8. Let AC equal the given diagonal and $\angle D = \angle B =$ the given angle; the angles adjacent to the diagonal are each equal to $\frac{1}{2}(180° - \angle B)$; construct isosceles triangles BAC and DAC on opposite sides of AC (§ 399, I).

9. Let AC equal the given diagonal and $AB = BC = CD = DA =$ the given side; construct isosceles triangles ACD and ACB on opposite sides of AC (§ 397); $ABCD$ is the required rhombus.

10. Let $AB = BC = CD = DA =$ the given side and $\angle BAC = \angle BCA = \angle DAC = \angle DCA =$ the given angle; construct isosceles triangles BAC and DAC on opposite sides of AC (§ 399, I)

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Construction Exercises. Circles

1. Construct a circle with perpendicular diameters; bisect one of the central angles of 90°; bisect one of the central angles of 45° forming a central angle of $22\frac{1}{2}°$; the sum of the adjacent central angles of 45° and $22\frac{1}{2}°$ is a central angle of $67\frac{1}{2}°$; its arc is the required arc.

2. Draw the diagonals of the given square; their intersection is the center of the required circle whose radius is the radius of the square.

3. Draw the diagonals of the given square; their intersection is the center of the required circle whose radius is the apothem of the square.

4. From the center of the given circle construct a line OA perpendicular to the given line; let OA intersect the circle at B; construct a line $BC \perp OA$; then BC is tangent to the circle (§ 207) and parallel to the given line (§ 98).

5. From the center of the given circle construct a line OA perpendicular to the given line x; construct radius $OB \perp OA$; at B construct $BC \perp OB$ then BC is tangent to the cricle (§ 207); $BC \parallel OA$ (§ 98); $BC \perp x$ (§ 99).

6. If OA is the perpendicular from the center to the required chord, construct chord $BC \perp OA$ at A; BC is the required chord.

7. If A is the mid-point of a required chord and O is the center of the circle, draw OA; construct chord $BC \perp OA$ at A; BC is the required chord.

8. Draw a line from O, the center of the given circle, to C, the center of the required circle; let OC intersect the given circle at A; then a circle with C as a center and CA as a radius is the required circle.

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Note: In numerous construction problems the magnitude to be found can be expressed as an algebraic relation to the given magnitudes. For example, if we are required to find a mean proportional to two given lines, a and b, we may say that we are to find a line x such that $x = \sqrt{ab}$. If we are required to find a square equal to a given triangle (whose base and altitude are respectively b and h), we may say we are to find $s^2 = \frac{1}{2}bh$ or $s = \sqrt{\frac{1}{2}b \cdot h}$.

Such interpretations of construction problems through algebraic expressions usually lead to one of three fundamental constructions. They are sometimes called the fourth proportional construction $\left(x = \frac{ab}{c}\right)$, the mean proportional construction $(x = \sqrt{ab})$, and the Pythagorean construction $(x = \sqrt{a^2 \pm b^2})$. Sometimes it may be necessary to make algebraic transformations to bring about the standard forms just given. For example, when $x^2 = a^2 - ab$ is changed to $x = \sqrt{a\,(a - b)}$ and when $x = a\sqrt{5}$ is changed to $x = \sqrt{5\,a \cdot a}$.

In general, if a construction problem can be expressed algebraically as:

1. the product of two lines divided by a third line, it may be solved by the fourth proportional construction;

2. the square root of the product of two lines, it may be solved by the mean proportional construction;

3. the square root of the sum (or difference) of the squares of two lines, it may be solved by the Pythagorean construction.

Occasionally it may be advisable to use the properties of other special figures. For example, in Ex. 6 since the altitude of an equilateral triangle equals one-half its side times $\sqrt{3}$, we may find the required line as suggested in the Hint, instead of transforming $x = a\sqrt{3}$ into $x = \sqrt{3}\,a \cdot a$ and using the mean proportional construction. In some problems involving $\sqrt{2}$, the properties of the square and its diagonals may be used.

The eight exercises on p. 335 should be thoroughly understood by pupils. Their mastery will facilitate the solutions of §§ 420, 421, 424 and numerous other problems which follow.

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Construction Exercises. Proportional Lines

1. x is a fourth proportional to a, b, and c.

2. x is a fourth proportional to $2\,a$, b, and c.

3. x is a fourth proportional to a and b.

4. x is a fourth proportional to a, b, and $2\,b$.

5. x is a mean proportional between a and b.

6. See the Hint.

7. x is a mean proportional between c and d.

8. x is the hypotenuse of a 45-45-90 triangle whose legs are equal to a.

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Construction Exercises. Equal Figures

1. Let the dimensions of the rectangles be a and b, x and b'; $ab = xb'$; x is a fourth proportional to b', b, and a.

2. Let the dimensions of the rectangles be a and b, $\frac{1}{2}\,a$ and x; $ab = \frac{1}{2}\,ax$; x is a fourth proportional to $\frac{1}{2}\,a$, a, and b.

3. Let s be a side of the given square, a an altitude of the triangle, and x its base; $s^2 = \frac{1}{2}\,ax$; x is a third proportional to $\frac{1}{2}\,a$ and s.

4. See the Hint.

5. See the Hint.

6. Use the method of Ex. 4 with $a = b$.

7. $x^2 = \frac{1}{2} h\,(b + b')$; x is a mean proportional between $\frac{1}{2} h$ and $(b + b')$.

8. HINT: The altitude of the required triangle equals the altitude of the given triangle and is the perpendicular bisector of its base.

9. Apply the Hint of Ex. 4 and check by § 322 except that Fact I becomes § 308.

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Construction Exercises. Circles and Regular Polygons

1. Inscribe a regular hexagon in the circle (§ 427); join the alternate vertices of the regular hexagon to form an equilateral triangle.

2. Find an equilateral triangle inscribed in the circle (Ex. 1); construct tangents to the circle at its vertices to form a circumscribed equilateral triangle (§ 335, 2).

3. Draw a circle with the given line as a radius; inscribe a regular hexagon.

4. Find the radius of the given circle; draw a circle with a radius three times as large.

5. Draw a circle with a radius equal to the sum of the radii of the given circles.

6. Let r and x be respectively the radii of the given and the required circles; $2\pi x = 2\pi r\sqrt{2}$; $x = r\sqrt{2}$; find x as in Ex. 6, p. 338, and with it describe a circle.

7. Draw a circle whose radius is twice the radius of the given circle; $\pi x^2 = 4\pi r^2$; $x^2 = 4r^2$; $x = 2r$.

8. If r is the radius of the given circle, draw a circle whose radius is $r\sqrt{3}$; $\pi x^2 = 3\pi r^2$; $x^2 = 3r^2$; $x = r\sqrt{3}$.

9. If a and b are radii of the given circles and c the radius of the required circle; $\pi c^2 = \pi a^2 + \pi b^2$; $c^2 = a^2 + b^2$; find c as in Ex. 4, p. 338.

10. A side of an inscribed equilateral triangle (Ex. 1) divides the circle into two arcs which have the ratio 1 : 2.

11. See Ex. 1.

12, 13, and 14. Inscribe a corresponding regular polygon in the circle (§§ 425, 427, 426) and construct tangents to the circle at the vertices of the inscribed polygon to form a circumscribed polygon of the required number of sides.

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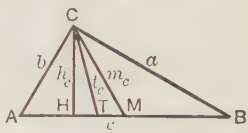
Supplementary Construction Exercises. Triangles

1. $AM = \frac{1}{2}C$; construct $\triangle ACM$ with $AC = b$, $MA = \frac{1}{2}c$, and $CM = m_c$; produce AM to B, making $AB = C$; draw CB; then $\triangle ABC$ is the required triangle.



2. Construct rt. $\triangle AHC$ with $AC = b$ and $CH = h_c$ (Ex. 4, p. 324); produce AH to B, making $AB = c$; draw BC.

3. Construct rt. $\triangle CHM$ with $CM = m_c$ and $CH = h_c$ (Ex. 4, p. 324); produce HM to A and B, making $AM = MB = \frac{1}{2}c$; draw AC and BC.

4. Construct rt. $\triangle CHM$ with $CM = m_c$ and $CH = h_a$ (Ex. 4, p. 324); with C as a center and a radius equal to b draw an arc intersecting MH produced at A; produce AM to B, making $MB = AM$; draw AC and BC.

5. Construct $\triangle ATC$ with $AC = b$, $CT = t_c$ and $\angle ACT = \frac{1}{2}\angle C$ (§ 398); construct $\angle TCB = \angle ACT = \frac{1}{2}\angle C$; the intersection of CB with AT produced is B; draw BC.

6. Draw $\triangle ATC$ with $CT = t_c$, $\angle ACT = \frac{1}{2}\angle C$, and $\angle CAT = \angle A$ (§ 399, Part II); draw $\angle TCB = \angle ACT = \frac{1}{2}\angle C$; produce a to intersect AT produced at B; draw BC.

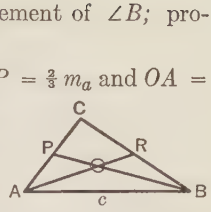
7. Find $\angle C$, given $\angle A$ and $\angle B$; proceed as in Ex. 6.

8. Construct rt. $\triangle CHT$ with $CT = t_c$ and $CH = h_a$ (Ex. 4, p. 324); construct $\angle ACT = \angle TCB = \frac{1}{2}\angle C$; produce a and b to meet HT produced at B and A.

9. Construct rt. $\triangle AHC$ with $CH = h_c$ and $\angle CAH = \angle A$ (Ex. 2,

p. 324); construct $\angle HCB$ equal to the complement of $\angle B$; produce a to intersect AH produced at B.

10. Let $BP = m_b$ and $AR = m_a$; $OB = \frac{2}{3} BP = \frac{2}{3} m_a$ and $OA = \frac{2}{3} AR = \frac{2}{3} m_a$ (§ 181); construct $\triangle OAB$ with $OA = \frac{2}{3} m_a$, $OB = \frac{2}{3} m_b$, and $AB = c$; produce OB to P making $BP = m_b$ and produce OA to R making $AP = m_a$; draw AP and BR and produce them to meet at a point, C.



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Exercises. Simple Loci

1. The locus is a circle with "your home" as a center and a one-mile radius (§ 437).

2. The locus is a circle with the same center and radius as the flywheel (§ 437).

3. The locus is a straight line parallel to the path of contact of the ball with the floor and the height of the radius above it.

4. The locus is two circles concentric with the given circle; if r is the radius of the fixed circle and r' is the radius of the rolling circle, $r + r'$ and $r - r'$ are the radii of the circles which are the locus (§ 438).

5. The locus is a circle concentric with the given circle; its radius equals a perpendicular from the center to one of the equal chords.

6. The locus is a circle concentric with the given circle; its radius equals one-half the radius of the given circle.

7. The locus is the perpendicular bisector of the line segment joining the two given points (§ 433).

8. The locus is the bisector of the angle (§ 434).

9. The locus is a line perpendicular to the given line at the given point.

10. The locus is the perpendicular bisector of the base of the rectangle (§ 433).

11. The locus is a circle with the hypotenuse of the right triangle as its diameter.

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Exercises. Compound Loci

1. The required points are the intersections of b and the circle whose center is D and whose radius is c; there are either two intersections, one intersection (if b is tangent to the circle), or no intersection.

2. The required points are the intersections of b with two lines $\parallel d$ and c units from d; these are two intersections unless b is parallel to the two lines, in which case there is no intersection.

3. The required point is the intersection of b with the perpendicular bisector (e) of CD. There is no intersection of $b \parallel e$.

4. The required points are the intersections of b with g and h, the bisectors of the angles formed by c and d. There are two intersections unless b is parallel to g or h, in which case there is no intersection.

5. The required points are the intersections of circle C with circle G whose center is G and radius is d. There are two intersections, one intersection (if the circles are tangent), or no intersection.

6. The required points are the intersections of circle C with g and h, parallel lines d units from line c. The number of intersections may be four, three, two, one, or none.

7. The required points are the intersections of circle C with d, a line parallel to the parallel lines g and h and midway between them. The number of intersections may be two, one, or none.

8. The required point is the intersection of the bisector of the given angle and an arc whose center is the vertex of the angle and whose radius is the given distance.

9. The center of the required circle is the intersection of b and the perpendicular bisector of CD. If the center is O, the radius $= OC = OD$.

10. The center of the required circle is the intersection of the bisectors of the angles formed by b and c with lines parallel to b and c and r units from them. There are four intersections and, therefore, any one of four circles may be the required circle.

11. The center of the required circle is the intersection of d and e, two lines $\parallel b$ and r units from b with a circle whose center is C and whose radius is r. If C is between d and e, there are two intersections; if C is without d and e and more than r units from either, the problem is impossible; otherwise there is one solution.

12. Draw $\triangle PRS$; through each vertex of the triangle, draw a line parallel to the opposite side; the intersections of these lines are the vertices of the required triangle. There is no solution if P, R, and S lie in a straight line.

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SUPPLEMENT

The New Type of Geometry Examination

The various Types (A to M) which follow are given merely as samples of some of the more common kinds of objective tests. Teachers should not use them as actual tests. As time goes on newer and better Types will be developed. Those given here, however, are fair examples of current tendencies.

Since the ability to plan and to make rigorous arguments is one of the most valuable ends in the teaching of geometry, no final test is sufficient which does not include full and formal demonstrations of standard book theorems and original exercises. What seems to be best to many teachers for monthly tests and final examinations is a combination of the old formal type of question (the subjective) with the new objective type of question.

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Type A. Statements to be Completed

1. figure, by, angle.

2. equal, right.

3. two equal.

4. arc, radii.

5. angles, arcs.

6. greater (less), less (greater), center.

7. semicircle.

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Type B. General Statements (Composition)

1. A shadow is a portion of a surface.

2. A square is a rectangle with a pair of intersecting sides equal.

3. A proportion is the equality between two equal fractions.

4. Euclid was born before Archimedes.

5. Vertical angles are always equal; adjacent angles may be equal.

6. An arch is a figure composed of an arc or of arcs.

7. A circle can be circumscribed about a rectangle, and no other parallelogram.

8. Descartes was a Frenchman.

10. A triangle two of whose sides are equal is isosceles.

11. The area of a regular polygon equals one-half its perimeter times its apothem.

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Type C. Conclusions from Figures

2. § 110.	**4.** § 224.	**5.** § 208.
6. § 321.	**8.** § 346.	**9.** § 167.
11. § 246.	**12.** § 257.	

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Type D. Converses (True, False, and Prose)

2. If two angles of a triangle are equal, the sides opposite are equal.

3. Two angles of 30° each are equal.

4. A rhombus is an equilateral quadrilateral.

5. Two chords equally distant from the center are equal.

6. Triangles with proportional sides are similar.

7. A regular polygon can be inscribed in a circle.

8. Two polygons which can be decomposed into the same number of like-placed similar triangles are similar.

9. If $\angle C$ is a right angle of $\triangle abc$, $a^2 + b^2 = c^2$.

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Type E. Definitions (Composition)

1. An equation is a statement of equality of two equal quantities.

2. The line joining two points of a circle is a chord.

4. Bisect means to cut into two equal parts.

5. A chord passing through the center of a circle is a diameter.

6. A straight angle equals two right angles.

7. The area of a surface is the number of square units of surface it contains.

10. An equilateral triangle is one whose sides are equal.

11. The length of a straight line between two points is the shortest distance between the points.

12. *Unity:* the number one, the ratio of two equal quantities; *external segment of a secant:* the part of the secant between some external point and its nearer point on the circle; *parenthesis:* either or both upright curves used to enclose several numbers which are to be taken as a single number; *non-consecutive:* not consecutive. For other definitions refer to the Index.

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Type F. Association (Statements and Facts)

1. The Facts in order are 2, 1, 4, 2, 5.

2. 4, 1, 1, 3, 5.

3. 1, 4, 4, 5.

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Type G. Quick Reasoning

Statements **2**, **4**, **6**, **8**, and **10** are correct.

11. $\angle 1 = \angle 2$ (§ 102); $\angle 5 = \angle 6$ (§ 101); $\overset{\frown}{TR} = \overset{\frown}{VS}$ (§ 232); $\angle 3 = \angle 7 = 90°$ (§ 226, *Prins.* 29, 26); $PT^2 = PR \times PS$ (§§ 314, 266); $TR^2 = PR \times RS$ (§§ 317, 266).

12. $\triangle$III $\sim$ $\triangle$I (§ 295); $\triangle AFB \sim \triangle$II (§ 295); $CF : CB = FD : AB$ (§ 292); there are seven similar triangles (§ 295) ($\triangle$ I, II, III, IV, *AFC*, *AFB*, *CAB*).

13. $\angle 7 + \angle 9 + \angle 4 + \angle 8 = 180°$ (§ 145).

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Type H. Quick Computations

Note: In questions of this type the answers only are required.

1. 10. **2.** 34°. **3.** $\frac{25\pi}{6}$; $\frac{25}{12}(2\pi - 3\sqrt{3})$. **4.** 45°, 45°, 90°; 36°, 72°, 72°. **5.** 45°. **6.** 60°, 6, $3\sqrt{3}$. **7.** $\overset{\frown}{TR} = 94°$, $\overset{\frown}{RS} = 116°$. **8.** $AE = 4(2 - \sqrt{2})$, $FE = 8(\sqrt{2} - 1)$. **9.** $BE = 25$, $AE = 20$. **10.** $8\sqrt{3}$. **11.** $20\sqrt{2}$, 20, $20\sqrt{3}$. **12.** 20, 9, 16.

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Type I. Locus Problems

1. The locus is a line parallel to the two given lines and midway between them.

2. The locus is the point of intersection of the two lines parallel to the given intersecting lines and one inch distant.

3. The locus is the points of intersection of the lines, two inches distant from the given line and parallel to it, with the circle, whose center is the given point and whose radius is two inches.

4. The locus is the points of intersection of the lines bisecting the two pairs of vertical angles formed by the given intersecting lines, with the lines two inches distant from the given line and parallel to it.

5. The center of the circumscribed circle is the point of intersection of the perpendicular bisectors of two sides of the given triangle.

6. The locus is a circle whose diameter is the line segment from the center of the given circle to the given point.

7. Let AB be the fixed hypotenuse whose mid-point is O; if $\triangle ACB$, $AC'B$, $AC''B$ are some of the triangles whose hypotenuse is AB, then $OA = OB = OC = OC' = OC''$ and the points A, B, C, C', C'' are equidistant from O and $\measuredangle C$, C', C'' are right angles. Therefore, the right-angled vertices lie on a circle whose center is O and whose radius is half the hypotenuse.

8. The locus is the points of intersection of two circles of radii three inches and two inches with the respective points as centers. Let l represent the distance between the given points. If $1 < l < 5$,

there are two points which comprise the locus; if $l = 1$ or $l = 5$, there is one point; if $l = 0$, $l < 1$, or $l > 5$, there are no points of intersection.

9. The locus is the lines parallel to the given base and at the altitude's distance from it.

10. The locus is the line perpendicular to the line of centers of the two given circles and passing through their point of contact.

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Type J. Association (Terms)

The selected terms in order are: rectangle, arc, chord, perpendicular, theorem, quadrilateral, point, polygon, bisector, denominator, perpendicular, parallelogram, line, diagonal, and equal.

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Type K. Interpreting Constructions

1. Radii of the same circle are equal.

2. Radii of equal circles are equal.

3. Along perpendiculars dropped from S to the sides.

4. Every point in the bisector of an angle is equidistant from the sides of the angle.

5. Corresponding parts of the congruent triangles OPS and OTS are equal.

6. It appears to be a regular hexagon.

7. They are mutually equiangular or their sides are proportional (the ratio is $\sqrt{3} : 1$).

8. $30° - 60° - 90°$ triangle.

9. Regular dodecagon.

10. They have equal central angles.

11. $AB = OC$, $BC = OC$ (§ 346); $OA = OC$ (*Prin.* 33); $AB = OC = BC = OA$ (Sub.); $ABCD$ is a $\square$ (§ 151).

12. $\triangle AKO \cong \triangle AKB$, $\triangle ABC \cong \triangle CDE$.

13. $\angle OPC = 120°$.

14. P and B are two points equidistant from O and C; they determine the perpendicular bisector of OC.

15. As in Ex. 14.

16. 7 and 11.

17. 1 and 4.

18. 2 and 9.

19. 3, 6, 8, and 10.

20. 1 is in the triangle, 2 is in the circle and the triangle, 3 is in the circle, 7 is in the rectangle and the circle, 11 is in the rectangle and the circle.

21. At m distance from AB.

22. At r distance from P.

23. Combine 21 and 22.

24. When the circle would be tangent to XY and would intersect ZW; when the circle would not intersect nor be tangent to XY and would intersect ZW or when the circle would be tangent to both XY and ZW; when the circle would be tangent to either XY or ZW and would not be tangent to or intersect the other; when the circle would neither be tangent to nor intersect either XY or ZW.

25. $RS = BG = O'H$, $OR = OS = O'H$.

26. To find the perpendicular bisector of OO' and thus determine the center of the circle passing through O and O'.

27. To make $\angle TO'P = \angle DBC = \text{rt. } \angle$.

28. It is inscribed in semicircle OBO'.

29. $BGHO'$ is a $\square$ and $\angle O'HG = \text{rt. } \angle$.

30. It is perpendicular to OG at G, to $O'H$ at H and it does not cross the line of centers.

31. Draw OA and AO'. Produce OA to intersect the larger given circle at some point, say K. At O' construct a line $\perp AO'$; let it cut the smaller given circle at L. Draw KL.

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Type L. Recognition (Multiple Choice)

1. A square and an equilateral triangle are regular polygons.

2. Equal complementary angles are of the same size and they are 45° angles.

3. Similar triangles are mutually equiangular and have their respective sides proportional.

4. In a parallelogram, the diagonals bisect each other and the opposite sides are equal.

5. In an isosceles right triangle the hypotenuse equals a side times $\sqrt{2}$ and the mid-point of the hypotenuse is equidistant from the vertices.

6. If the angles of a triangle have the ratio of 1 : 2 : 3, one angle is a right angle and one side is double another.

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Type M. Recognition (Analogy)

1. Rhombus. **2.** Surface. **3.** arc degree. **4.** s^2. **5.** Radius. **6.** The radius of the inscribed circle. **7.** Perimeter.

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Suggestion: The Word List can be made the basis for spelling lessons, spelling matches, or for discussions of the uses of the various terms and the theorems in which the words occur.

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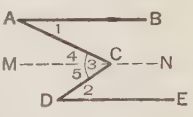
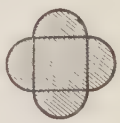
General Miscellaneous Exercises

1. Through C draw $MN \parallel AB$ (*Prin.* 14); $\angle 1 = \angle 4$ (§ 101); $\angle 1 + \angle 2 = \angle 4 + \angle 5$ (Given); $\angle 2 = \angle 5$ (*Prin.* 6); $MN \parallel DE$ (§ 114); $AB \parallel DE$ (*Prin.* 16).

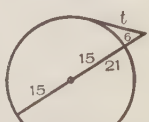


2. 25 sq. in. = area of square; $2\,(6.25\,\pi)$ sq. in. = area of the four semicircles; $12.5\,(2 + \pi)$ = total area.



3. The garage is at the intersection of the perpendicular bisectors of two of the sides of $\triangle ABC$.

4. Secant $\times$ Ext. Seg. $= 36 \times 6$. The tangent $= \sqrt{36 \times 6} = 6\sqrt{6}$.

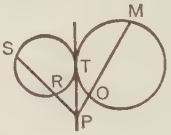


5. $\angle ODE$ = numerically $\frac{1}{2}(\overset{\frown}{OG} + \overset{\frown}{CB})$; $\angle OEC$ = numerically $\frac{1}{2}\,(\overset{\frown}{OC} + GA)$ (§ 229); $\angle ODE = \angle OEC$ (Sub.); $\triangle ODE$ is isos. (§ 120).

6. CO produced passes through the center of the circle (§ 202); call the remaining segment of the diameter OX; $OX \cdot OC = AO \cdot OB$ (§ 313); $OX = \frac{8 \times 8}{4} = 16$; radius $= \frac{16 + 4}{2} = 10$.

7. Let $2R = x + h$, where x is the remaining segment of the diameter; $xh = \left(\frac{s}{2}\right)^2$ or $x = \frac{\left(\frac{s}{2}\right)^2}{h}$ (§ 313); $R = \left[\frac{\left(\frac{s}{2}\right)^2}{h} + h\right] \div 2$ (Sub.).

8. $PT^2 = PS \times PR$; $PT^2 = PM \times PO$ (§§ 314, 226); $PS \times PR = PM \times PO$ (Sub.).

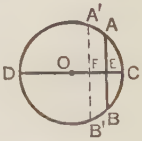


9. $\angle C = \angle ABC$, $\angle BED = \angle D$ (§ 80); $\angle ABC = 2\angle D = \angle C$ (§ 110, Sub.); $\angle FEC = \angle BED = \angle D$ (*Prin.* 27, Sub.); $\angle AFD = \angle C + \angle FEC = 2\angle D + \angle D = 3\angle D$ (§ 110, Sub.).

10. rt. $\triangle BDC \cong$ rt. $\triangle CEB$ (*hy.a.* = *hy.a.*); $DE \parallel CB$ (§ 247).

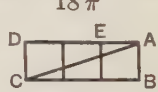
11. $\angle 1 + \angle 2 = \angle 3 + \angle 4$ (Given); $h \perp BA$ (Given); $\angle A = \angle 4$ (*Prin.* 28); $CM = AM = BM$ (Ex. 15, p. 120); $\angle A = \angle 1$ (§ 80); $\angle 4 = \angle 1$ (Sub.); $\angle 2 = \angle 3$ (*Prin.* 6).

12. Let $OF = FE = EC = x$. (1) $EC = x$; $ED = 5x$; $AB = 5$; $AE = EB = 2\frac{1}{2}$. $2\frac{1}{2} \times 2\frac{1}{2} = x \cdot 5x$ (§ 313); $x = \frac{1}{2}\sqrt{5}$; $OC = \frac{3}{2}\sqrt{5}$. (2) $FC = 2x$; $FD = 4x$; $A'F = FB' = 2\frac{1}{2}$. $2\frac{1}{2} \times 2\frac{1}{2} = 2x \cdot 4x$ (§ 313); $x = \frac{5}{8}\sqrt{2}$; $OC = \frac{15}{8}\sqrt{2}$.



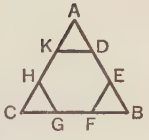
13. Equal lengths of belt must travel over each of the two pulleys in a given time. $(24\pi \cdot 360)$ in. $= (18\pi \cdot s)$ in.; $s = \frac{24\pi \cdot 360}{18\pi} =$ 480 R.P.M.

14. 8 $AB = 80$; $AB = 10$; Square $EB = 100$; $AC = \sqrt{(10)^2 + (30)^2} = \sqrt{1000} = 10\sqrt{10}$.

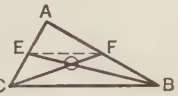


15. $\angle A = \angle B = \angle C = 60°$ (§§ 81, 106); $\angle ADK = \angle AKD$ (§ 80); $\angle ADK = 60°$ (§ 106, Sub.); $\angle KDE = 120°$ (*Prin.* 29); $AK = AD = KD$ (§ 121); $\triangle AKD \cong \triangle BEF \cong \triangle GCH$ (*s.a.s.* =

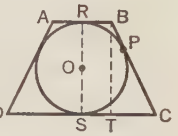
s.a.s.); $KD = DE = EF =$ etc. (Sub.); $KDEFGHK$ is a regular hexagon (Def.); $\triangle ADK \sim \triangle ABC$ (*a.a.* = *a.a.*); $AD = \frac{1}{3} AB$ (Given); $\triangle ADK = \frac{1}{9} \triangle ABC$ (§ 308); $\triangle ADK + \triangle BEF + \triangle CGH = \frac{1}{3} \triangle ABC$ (*Prin.* 5); $KDEFGHK = \frac{2}{3} \triangle ABC$ (*Prin.* 6).



16. Draw EF (*Prin.* 13); $EF \parallel CB$ (§ 155); $\triangle FBC = \triangle EBC$ (§ 246); $\triangle OBC = \triangle OBC$ (Iden.); $\triangle FOB = \triangle EOC$ (*Prin.* 6); $\triangle BAE = \triangle BEC$ (§ 245); $AEOF = \triangle BCO$ (*Prin.* 6).

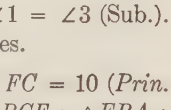


17. $AB = 6$, $DC = 18$ (Given); $RB = BP = 3$, $SC = CP = 9$ (§ 211, Sub.); $BC = 12$, $TC = 6$ (*Prins.* 5, 6); $BT = 2$ radii $= \sqrt{12^2 - 6^2} = 6\sqrt{3}$ (§ 321); Radius $= 3\sqrt{3}$ (*Prin.* 8).



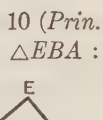
18. (*a*) $\angle 1 = \angle 2$ (*Prin.* 28). (*b*) By Ex. 16, p. 120, using the mid-point of OC as a center, a circle may be passed through P, O, S, and C. (*c*) $\angle 1 = \frac{1}{2} \overset{\frown}{OS}$, $\angle 3 = \frac{1}{2} \overset{\frown}{OS}$ (§ 224); $\angle 1 = \angle 3$ (Sub.). (*d*) $\angle 4 = \angle 2$ (Part c); $\angle 3 = \angle 4$ (Sub.). (*e*) Yes.

19. Draw $BF \parallel AD$; $BF = 8$, $DF = 20$ (§ 147); $FC = 10$ (*Prin.* 6); $\triangle BCF \sim \triangle EBA \sim \triangle ECD$ (*a.a.* = *a.a.*); $\triangle BCF : \triangle EBA : \triangle ECD = 10^2 : 20^2 : 30^2 = 1 : 4 : 9$. $\triangle BCF = \sqrt{12 \cdot 2 \cdot 6 \cdot 4} = 24$ (§ 326); $\triangle ECD = 9 \times 24 = 216$, $\triangle EBA = 4 \times 24 = 96$; $ABCD = 216 - 96 = 120$.

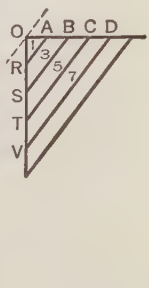


20. $\overset{\frown}{BX} = \overset{\frown}{AY}$, $\overset{\frown}{CX} = \overset{\frown}{AZ}$ (§ 232); $\overset{\frown}{BC} = \overset{\frown}{YZ}$ (*Prin.* 6); $\angle ZBY = \angle BYC$ (§ 224, Sub.); $BZ \parallel CY$ (§ 114).

21. $\angle 3 = \angle 2 = 90°$ (Given); $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$ (§ 164); $\angle 1 + \angle 4 = 180°$ (*Prin.* 6).

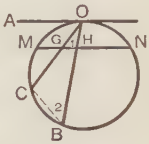


22. $OA = AB = BC = CD$ (§ 282); $OB = 2\,OA$ (*Prin.* 4); $\triangle OAR : \triangle OBS = (OA)^2 : (2\,OA)^2$ (§ 308, Sub.); $\triangle OBS = 4\,\triangle OAR$ (§ 266); $\triangle OBS = \triangle OAR + ABSR = 4\,\triangle OAR$ (*Prin.* 4, Sub.); $ABSR = 3\,\triangle OAR$ (*Prin.* 6); in like manner, $BCTS$ and $CDVT$ may be shown to equal respectively $5\,\triangle OAR$ and $7\,\triangle OAR$.



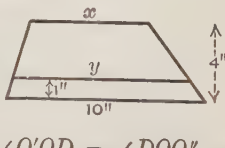
23. $OD = OA = OG = OK = 6''$; $OM = MN = ND = 2''$; $BM \times MF = BM^2 = MD \times MK = 4 \times 8 = 32$ (§ 313, Sub.); $BM = 4\sqrt{2}$; $CN = \sqrt{2 \times 10} = 2\sqrt{5}$; $\triangle CDE = \frac{1}{2} \times 2 \times 2 \times 2\sqrt{5} = 4\sqrt{5}$; $CEFB = \frac{1}{2} \times 2 \times (4\sqrt{5} + 8\sqrt{2}) = 4\sqrt{5} + 8\sqrt{2}$; $BFGA = \frac{1}{2} \times 2 \times (8\sqrt{2} + 12) = 8\sqrt{2} + 12$; Polygon $ABCDEFG = 4\sqrt{5} + 4\sqrt{5} + 8\sqrt{2} + 8\sqrt{2} + 12 = 8\sqrt{5} + 16\sqrt{2} + 12 = 4(2\sqrt{4} + 4\sqrt{2} + 3)$; the approximate area of the circle is $8(2\sqrt{5} + 4\sqrt{2} + 3)$.

24. $\angle 1$ = numerically $\frac{1}{2}(\overset{\frown}{ON} + \overset{\frown}{MC})$ (§ 229); $\angle 2$ = numerically $\frac{1}{2}(\overset{\frown}{OM} + \overset{\frown}{MC})$ (§ 224); $\overset{\frown}{ON} = \overset{\frown}{MC}$ (§ 232); $\angle 1 = \angle 2$ (Sub.); $\angle O = \angle O$ (Iden.); $\triangle OGH \sim \triangle OBC$ (*a.a.* = *a.a.*); $OC : OH = OB : OG$ (§ 292); $OC : OB = OH : OG$ (§ 272).



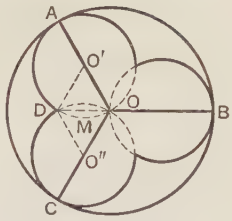
25. Let R = radius of the circle. $OM = \frac{1}{2}R$ (§ 158); $EM = \frac{1}{2}R\sqrt{3}$ (§ 329); $EC = R\sqrt{3}$; $\triangle ACE = \left(\frac{R\sqrt{3}}{2}\right)^2\sqrt{3} = \frac{3R^2}{4}\sqrt{3}$ (§ 327); $FE = R$ (§ 346); $ON = \frac{1}{2}R\sqrt{3}$ (§ 327); Hexagon $= \frac{1}{2} \times \frac{1}{2}R\sqrt{3} \times 6R = \frac{3R^2}{2}\sqrt{3}$ (§ 349); $\overset{\frown}{CDE} = 120°$ (§ 346); $\angle ECH = \angle HEC = 60°$ (§ 219); $\angle H = 60°$ (§ 106); $HC = HE = EC = R\sqrt{3}$ (§ 121); $KH = 2R\sqrt{3}$; $\triangle GHK = \left(\frac{2R\sqrt{3}}{2}\right)^2\sqrt{3} = 3R^2\sqrt{3}$ (§ 327); $\triangle GHK$: hexagon $= 3R^2\sqrt{3} : 3\frac{R^2\sqrt{3}}{2} = 1 : \frac{1}{2}$ and hexagon : $\triangle ACE = \frac{3R^2\sqrt{3}}{2} : \frac{3R^2\sqrt{3}}{4} = 1 : \frac{1}{2}$ (Sub. and § 42, *C*); $\triangle GHK$: hexagon = hexagon : $\triangle ACE$ (Sub.).

26. By § 257 and Sub., $32 = \frac{1}{2} \cdot 4(10 + x)$ or $x = 6$; $32 = \frac{1}{2} \cdot 3(6 + y) + \frac{1}{2} \cdot 1(y + 10)$ or $y = 9$.

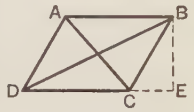


27. $\triangle OO'D \cong \triangle OO''D$ (*s.s.s.* = *s.s.s.*); $\angle O'OD = \angle DOO'' = 60°$ (*Prins.* 38 and 8); $\angle O'OD = \angle ODO' = \angle DO'O = 60°$ (§§ 79, 121); $\overset{\frown}{OMD} = 60°$ (*Prin.* 31); there are six such dotted arcs in the

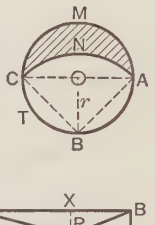
figure. The circumference of trefoil ABC $= 3\,(2\,R\pi) - 6$ arcs $= 3\,(2\,R\pi) - 6 \cdot \frac{1}{6} \cdot 2\,R\pi = 2 \cdot 2\,R\pi = 4\,R\pi =$ the circumference of circle O. The area of trefoil $ABC = 3\,\pi R^2 - 6$ (segment OMD). Segment OMD $=$ sector $O'OMD - \triangle O'OD = \frac{1}{6}\,\pi^2 \left(\frac{R}{2}\right) - \left(\frac{R}{2}\right)^2 \sqrt{3} = \frac{\pi R^2}{24} - \frac{R^2\sqrt{3}}{4}$. Trefoil $= 3\,\pi R^2 - \frac{\pi R^2}{4} + \frac{3\,R^2\sqrt{3}}{2} = \frac{11\,R^2}{4}\pi + \frac{6\,R^2}{4}\sqrt{3} = \frac{R^2}{4}\,(11\,\pi + 6\sqrt{3})$.



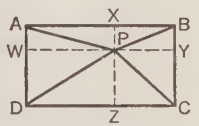
28. $AD = 7$, $DC = 9$, $AC = 8$ (Given); $\triangle ADC = \triangle BCD$ (§ 246); $\sqrt{12 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{2}\,BE \cdot 9$ (§§ 326, 242); or $BE = \frac{8}{3}\sqrt{5}$; $CE = \sqrt{(7)^2 - (\frac{8}{3}\sqrt{5})^2} = \frac{1}{3}$; $BD = \sqrt{(\frac{8}{3}\sqrt{5})^2 + (\frac{3}{3})^2} = 14$ (§ 321).



29. $AB = r\sqrt{2}$; $\overset{\frown}{CMA} = \frac{1}{2} \cdot 2\,r \cdot \pi = r\pi$; $\overset{\frown}{CNA} = \frac{1}{4} \cdot 2\,r\pi\sqrt{2} = \frac{r\pi\sqrt{2}}{2}$; Circumference of $CMANC = \pi r\,(1 + \frac{\sqrt{2}}{2})$. Area of $CMANC = \frac{1}{2}\,\pi r^2 - (\frac{1}{4}\,\pi (r\sqrt{2})^2 - r^2) = r^2$.



30. Draw $XZ \perp DC$ and $WY \perp BC$; $XZ \perp AB$, $WY \perp AD$ (§ 99); $(PA)^2 = (PX)^2 + (XA)^2$, $(PC)^2 = (PZ)^2 + (ZC)^2$, $(PB)^2 = (PY)^2 + (YB)^2$, $(PD)^2 = (PW)^2 + (WD)^2$ (§ 321); $(PA)^2 + (PC)^2 = (PX)^2 + (XA)^2 + (PZ)^2 + (ZC)^2$, $(PB)^2 + (PD)^2 = (PY)^2 + (YB)^2 + (PW)^2 + (WD)^2$ (*Prin.* 5); $(PX)^2 = (BY)^2$, $(XA)^2 = (PW)^2$, $(PZ)^2 = (WD)^2$, $(ZC)^2 = (PY)^2$ (§ 147, *Prin.* 9); $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$ (Sub.).

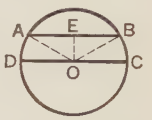


31. Draw DB and $D'B \cdot \angle BAD = \angle GAC$ (Iden.); rt. $\triangle ADB \sim$ rt. $\triangle ACG$ (§ 295); $\angle BAD' = \angle G'AC$ (Iden.); rt. $\triangle AD'B \sim$ rt. $\triangle ACG'$ (§ 295); $AB : AG = AD : AC$ and $AB : AG' = AD' : AC$ (§ 292); $AD \times AG = AB \times AC = AD' \times AG'$ (§ 266, Sub.).

32. The shaded portion equals the square minus two semicircles plus two segments $= (2R)^2 - 2\frac{\pi R^2}{2} + 2\left(\frac{1}{4}R^2\pi - \frac{R^2}{2}\right) = 4R^2 - \pi R^2 + \frac{1}{2}\pi R^2 - R^2 = 3R^2 - \frac{\pi R^2}{2} = \frac{R^2}{2}(6 - \pi)$.

33. $(AB)^2 = (AC)^2 + (BC)^2$, $(BC)^2 = (BD)^2 - (DC)^2$ (§ 321); $(AB)^2 = (AC)^2 + (BD)^2 - (DC)^2$ (Sub.); $(BD)^2 + (AC)^2 = (AB)^2 + (DC)^2$ (*Prins.* 5, 6).

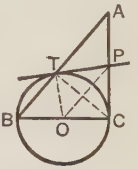
34. $OB = 6$, $OE = 3$, $OE \perp AB$; $\angle BOE = 60°$ (§ 159); $EB = 3\sqrt{3}$ (§ 329); $\triangle OEB = \frac{1}{2} \cdot 3 \cdot 3\sqrt{3} = \frac{9\sqrt{3}}{2}$; sector $OBC = \frac{1}{12} \cdot 36\pi = 3\pi$; $ABCD = 2\left(\frac{9\sqrt{3}}{2} + 3\pi\right) = 9\sqrt{3} + 6\pi$; grass plot $= 36\pi - 9\sqrt{3} - 6\pi = 3(10\pi - 3\sqrt{3})$.



35. Circumference $= 2r\pi = 360°$; required arc $= r = x°$; $x = \frac{360}{2\pi} = \frac{180}{\pi} = 57.2958°$.

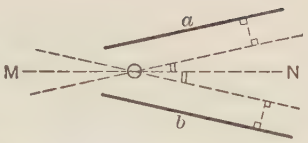
36. The white segments $= \frac{1}{2}\left(\frac{c}{2}\right)^2\pi - \frac{ab}{2}$; $A + B = \frac{1}{2}\left(\frac{a}{2}\right)^2\pi + \frac{1}{2}\left(\frac{b}{2}\right)^2\pi - \left[\frac{1}{2}\left(\frac{c}{2}\right)^2\pi - \frac{ab}{2}\right]$; $A + B = \frac{a^2}{8}\pi + \frac{b^2}{8}\pi - \frac{c^2}{8}\pi + \frac{ab}{2}$; $A + B = \frac{ab}{2}$ (since $a^2 + b^2 = c^2$); $C = \frac{ab}{2}$; $A + B = C$.

37. $OT = OC$ (*Prin.* 33); $PT = PC$ (§ 211); $PO \perp TC$ (§ 171); $BA \perp TC$ (§ 226); $PO \parallel BA$ (*Prin.* 17); $BO = OC$ (*Prin.* 33); $AP = PC$ (§ 154).

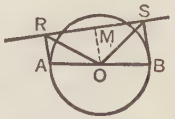


38. $AB = 12''$, $FD = 4''$; $FD \times FG = (FC)^2$ (§ 313); $4 \times 8 = FC^2$; $FC = 4\sqrt{2}$. The area of the surface is the area of a circle whose radius is $4\sqrt{2} = 32\pi$.

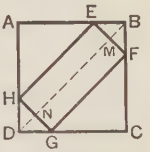
39. At equal distances from the given lines a and b construct two lines parallel to a and b respectively and intersecting at O. MN, the bisector of $\angle O$, is the required line.



40. Draw $OM \perp RS$. $AR \perp RS$, $BS \perp RS$ (Given); $RA \parallel OM \parallel BS$ (*Prin.* 17); $AO = OB$ (*Prin.* 33); $RM = MS$ (§ 282); rt. $\triangle OMR \cong$ rt. $\triangle OMS$ (*l.l.* = *l.l.*) $OR = OS$ (*Prin.* 38).

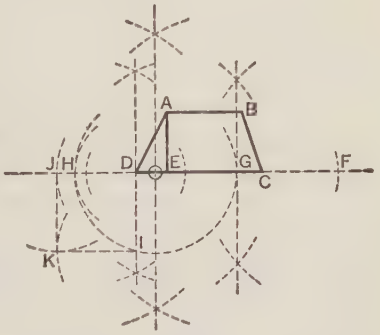


41. $EM = MB = 2.5$; $DN + MB = 5$; $AB = AD = 12$; $DB = 12\sqrt{2}$ (§ 251); $MN = 12\sqrt{2} - 5 = 11.968$.



42. $\overset{\frown}{TB} = \overset{\frown}{DC}$ (§ 232); $\sim \angle ATB =$ numerically $\frac{1}{2}\overset{\frown}{TB}$ (§ 219); $\angle CTD =$ numerically $\frac{1}{2}\overset{\frown}{DC}$ (§ 224); $\angle ATB = \angle CTD$ (Sub.); $\angle A =$ numerically $\frac{1}{2}(\overset{\frown}{DC} + \overset{\frown}{DT} - \overset{\frown}{TB}) =$ numerically $\frac{1}{2}\overset{\frown}{DT}$ (§ 230); $\angle DCT =$ numerically $\frac{1}{2}\overset{\frown}{DT}$ (§ 224); $\angle A = \angle DCT$ (Sub.); $\triangle ABT \sim \triangle CDT$ (*a.a.* = *a.a.*).

43. Let AE be the altitude of the given trapezoid $ABCD$. We wish to find the square of side $s = \sqrt{\frac{1}{2}AE\ (AB + DC)}$. Make $CF = AB$. Bisect DF in G. Make $DH = AE$. Bisect HG in O. Describe a semicircle with O as center and HG as diameter. Erect a perpendicular at D which intersects the semicircle at I. Make DJ, JK, and IK equal to DI. $JDIK$ is the required square.



44. $\dfrac{\text{Given circle}}{\text{Required circle}} = \dfrac{2}{1} = \dfrac{R^2}{r^2}$ or $r = \sqrt{R \cdot \dfrac{R}{2}}$; the required radius ($r$) is the mean proportional between the given radius (R) and half the given radius $\left(\dfrac{R}{2}\right)$.

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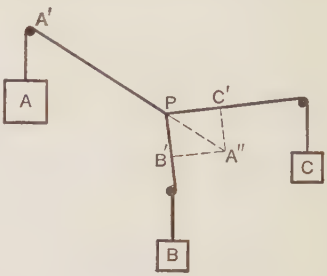
Applications of Geometry

1. The longer strip should be the perpendicular bisector of the shorter.

2. There are two equal segments of the cord ($l.l. = l.l.$); yes; because the four triangles formed are rigid figures.



3. $\angle B'PC' = 90°$; $B = C$; $A = B\sqrt{2} = C\sqrt{2}$.

4. A rectangle may change its shape without changing the lengths of its sides.

5. $AB = DC$ and $AD = BC$.

6. Drop a perpendicular from A to the edge of the cushion in which P is to be found. Produce this line its own length. From the extremity, draw a line to pocket C. P is the intersection of this line and the inner edge of the cushion.

7. He ran another from B to D; the intersection of the cords is the center of the ceiling.

8. $BC = 100$ yds.; $\angle A = 20°$ (§ 110); $BA = 100$ yds. (§ 120).

9. $\angle D$ = rt. $\angle$ (Given); $AD = AD'$ (Given); $AD'ED$ is a square (Def.).

10. The angle to be cut at A is 30°; the angle at B is 60° (§ 159).

11. $BC \parallel OA$ (*Prin.* 17).

12. $\triangle AED \cong AEC$ (*s.s.s.* = *s.s.s.*). Yes, since $\angle DAE = \angle CAE$ (*Prin.* 38).

13. The carpenter should find if $AB = ED$, $AC = EC$, and $BC = DC$.

14. $BP \perp AP$ (§ 226).

15. B will revolve faster than A; B makes $1\frac{2}{3}$ revolutions to one of A; B makes 300 R. P. M.; the speed is not changed but the motion of gear B is reversed; C makes 360 R. P. M.

16. If gear A is the follower and is larger than B it produces more power; A makes $\frac{3}{8}$ of a revolution to one of B; B makes 72 R. P. M.

17. There is a greater amount of rolling friction and a better chance for lubrication.

18. The circle which surrounds the three islands and passes through A and B has an inscribed angle such as $\angle ACB$, all other angles inscribed in the same segment are equal to 40°, and any angle, whose vertex is without the segment and whose sides pass through A and B, is less than 40°.

19. Since $\overset{\frown}{AB} = \frac{1}{10}$ the circumference of the circle, it equals $\frac{1}{10}$ of 360° = 36° and $\angle AOB = 36°$ (*Prin.* 31).

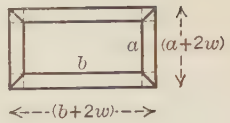
21. PB is a part of the diameter by § 202. (*a*) The intersection of PB with the perpendicular bisector of any other chord will determine the center. (*b*) If BP is produced through P to intersect the circle in C and PC is measured, PC times the remainder of the diameter = 36. The radius may then be found and if measured from C and along CPB, the center may be found.

22. The complete area $= 2\,(\frac{1}{2} \cdot 2 \cdot 4) + 2\,[\frac{1}{2} \cdot 2\,(4 + 6)] + 2\,[\frac{1}{2} \cdot 2\,(6 + 8)] + 2\,[\frac{1}{2} \cdot 2\,(8 + 9)] = 2\,(4 + 4 + 6 + 6 + 8 + 8 + 9) = 2 \cdot 45 = 90$.

23. The figure formed by the opposite edges of the cloth and the two positions of the tape is a parallelogram. This may be proved by drawing a diagonal and proving two triangles congruent by *s.a.s.* = *s.a.s.* The segments of the edges of the cloth between the ends of the tape are equal by *Prin.* 38. The figure is a parallelogram by § 151. There are five parallelograms formed by § 152 (a). The edges of the cloth and the dotted lines are parallel by the definition of a parallelogram and *Prin.* 16. The width of the cloth is divided into five equal parts by § 282.

24. The number of feet of baseboard required equals $18 + 11 + 8 + 2 + 10 + 3 = 52$. The perimeter of the room shown is the same as that of a room $11' \times 18'$. The area of the room shown ($18 \times 6 + 8 \times 5 = 148$) is 50 sq. ft. less than that of a room $11' \times 18'$.

25. Since picture frames are usually mitered at the corners the length of the molding required may be found from the outside perimeter of the frame which is: $2\,(a + 2\,w) + 2\,(b + 2\,w) = 2\,(a + b + 4\,w)$.



26. $\dfrac{50\,(40 + 70)}{2 \times 300} = 9\frac{1}{6}$; one would buy 10 qts.

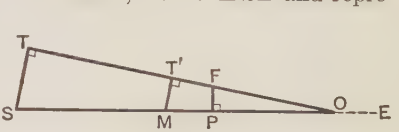
27. The surface to be painted equals $(2 \times 9 \times 8) + 2 \times 15 \times 8 + 2 \times \frac{1}{2} \times 9 \times 3 = 144 + 240 + 27 = 411$ sq. ft. $411 \div 200 = 2\frac{11}{200}$ gal. $= 2$ gal. $\frac{1}{2}$ pt.

28. CAB is a straight line ($\angle DAC + \angle DAH + \angle HAB =$ st. $\angle$); $\triangle CDA \sim \triangle CGB \sim \triangle AHB$ (*a.a.* = *a.a.*); $\dfrac{CB}{AB} = \dfrac{BG}{BH} = \dfrac{3}{1}$ (§ 292 and Sub.); $CB = 3\,AB$ (§ 266).

29. $\triangle OAB \sim \triangle OCD$ (§ 300); $CD : AB = OC = OA$ (§ 292); the distance between C and D is $\frac{5}{3}$ of the distance between A and B.

30. $AB = 20'$, $OD = 6''$, and $A'B' = 2''$. $6 : 2 = OD : 20$. The altitudes of similar triangles have the same ratio as any two corresponding sides.

31. Let S, M, and E represent the centers of the sun, moon, and earth. Since the width of the path of darkness is only a small fraction of the circumference of the earth, let $PF \perp SE$ and represent one-half the width of the path. Assume the following to be known; TS = radius of the sun, MT' = the radius of the moon, PE = the radius of the earth, and MP = distance from the center of the moon to the nearest point on the earth. SO and MO may be calculated by using proportional parts of similar triangles, OTS and $OT'M$. $PO = MO - MP$. $T'O = \sqrt{(MO)^2 - (MT')^2}$. $T'O : PO = MT' : FP$ from which FP may be calculated. The width of the path is approximately equal to $2\,FP$.



32. Let x = no. of ft. in the height of the pole; $x + 1$ = no. of ft. in the length of the rope; $(x + 1)^2 = x^2 + 11^2$; $x^2 + 2\,x + 1 = x^2 + 121$; $2\,x = 120$; $x = 60$ ft.

33. Construct a right angle C'; lay off on its sides $C'A' = 5'$ and $C'B' = 12'$. The hypotenuse $= 13'$ (§ 321); $\triangle A'B'C' \cong \triangle ABC$ (*s.s.s.* = *s.s.s.*); $\angle C = 90°$ (*Prin.* 38 and Sub.).

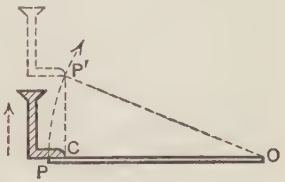
Note: This solution is derived from the one employed to prove the converse of the Pythagorean Theorem.

34. $\frac{AB}{2}\sqrt{3} = 10$ (§ 327); $AB = \frac{20}{3}\sqrt{3} = CD = CB$; $AD = 2\,AB = \frac{40}{3}\sqrt{3}$.

35. He should first find the line ADB and at D erect a perpendicular to it. On this perpendicular he should take any point such as C and measure $\angle DCB$. Then he should turn an angle of 90° to the right and locate the point (such as A) of intersection of this new line (CA) and the original line ADB. By measuring AD and DC and applying § 317, DB may be found. ($AD : CD = CD : DB$.)

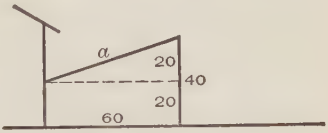
36. $\angle 3 = \angle 4$ (the angle of incidence of reflected light equals the angle of reflection); $\angle 1 = \angle 2$ (*Prin.* 28); a, b, and c can be measured; $a : x = b : c$ (§ 292); $x = ac \div b$.

37. $\triangle BFE \sim \triangle BCA$ (*a.a.* = *a.a.*); let $EF = CF = x$; $x : 21 = 28 - x : 28$ (§ 292); $28\,x = 21\,(28 - x)$ or $x = 12$.

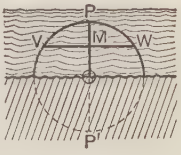


38. $OP = OP' = 13''$; $PC = 1''$; $OC = 12''$; $P'C = 5''$ (§ 321).

39. $a^2 = 60^2 + 20^2$, or $a = 20\sqrt{10} = 63.24$; $63.24 + 6. = 69.24$; he would buy 70 ft. of wire.



40. $PM \times P'M = VM \times MW$ (§ 313); $4 \times 16 = VM^2$; $VM = 8$ and $VW = 16$. Time = Distance ÷ Rate = 16 ÷ 20 = $\frac{16}{20}$ hours or 48 minutes.



41. $2 : 1 = x^2 : 4^2$ (§ 364 and Given); $x = 4\sqrt{2} = 5.656$. Approximately 1′ 8″ should be added to the diameter.

42. The area of the circle = 452.39; the area of the square = $[(\frac{8}{4} \cdot 12)\sqrt{2}]^2 = 450$.

43. $OB : OA = 8 : 1$ = distance B moves : distance A moves (§ 292); or A moves $\frac{1}{16}''$ when B moves $\frac{1}{2}''$. A and B each turn through $4°$.

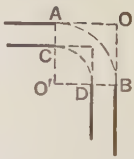
44. $10 : 3 = 60 : x$ or $x = 18$ lbs.

45. See Statement 6, § 363.

46. The area of the circle $= 81 \times 355 \div 113 = 254.469026$ (using the value of π given by Metius, § 361). By the Egyptian method the area $= (\frac{8}{9} \cdot 18)^2 = 16^2 = 256$. The error $= 1.530974$ sq. in. Area $= \pi r^2 = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$. By the Egyptian method, the area $= \frac{64}{81} D^2$. $\frac{\pi}{4} = \frac{64}{81}$; $\pi = 4 \times 64 \div 81 = 3.1604+$. See also Ex. 13, p. 295.

47. $BC = \frac{1}{2} R$ (§ 329); $CA = OA - OC = R - \frac{1}{2} R\sqrt{3}$ (§§ 346, 327); $BA = \sqrt{\left(\frac{R}{2}\right)^2 + \left(R - \frac{1}{2} R\sqrt{3}\right)^2}$ (§ 321); $BA = \sqrt{\frac{R^2}{4} + R^2\left(\frac{7}{4} - \sqrt{3}\right)} = R\sqrt{2 - \sqrt{3}}$. This result may also be found by § 353. The central angle between any two adjacent spokes is $\frac{1}{12} \times 360° = 30°$. Twelve equally spaced points may be located as follows: (1) inscribe a regular hexagon and bisect its central angles; or (2) construct a central angle of $30°$ and take its chord twelve times around the circle.

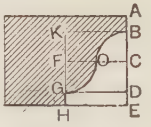
48. The centers of the arcs is point O'; $AOBO'$ is a square; $O'B = 30'$ and $O'D = 15'$. $\overset{\frown}{AB} - \overset{\frown}{CD} = \frac{1}{4} \cdot 60 \cdot \pi - \frac{1}{4} \cdot 30 \cdot \pi = \frac{30\pi}{4}(2 - 1) = \frac{15\pi}{2} =$ 23.55.



49. Diagonals AC and BD intersecting at O are drawn. With A as a center and with radius AO the arc TON is drawn; in like manner arcs MOQ, POS, and ROU are drawn. Let $AB = s$, then $OB = MB = AN = \frac{s}{2}\sqrt{2}$; $AM = AU = NB = s - \frac{s}{2}\sqrt{2}$; $UM = s(\sqrt{2} - 1)$. $MN = AB - AM - NB = s - 2\left(s - \frac{s}{2}\sqrt{2}\right) =$

$s(\sqrt{2}-1)$. In like manner, the octagon may be shown to be equilateral and since each angle (such as $\angle UMN$) equals 135°, the figure $UMNPQRSTU$ is a regular octagon.

50. AE is divided into 6 equal parts; $KBEH$ is a rectangle with $KB = 2\,BC$; $\overset{\frown}{BO}$ is a quadrant with center C and radius BC; $\overset{\frown}{OG}$ is a quadrant with center F and radius OF.



51. Bisect the angle at A and erect a perpendicular to AB at B which intersects the bisector of $\angle A$ at O. BC is the arc of a circle with center O and radius OB.

52. The gardener set out the plants along the dotted line represented in the annexed figure. The arc at the corner is a quadrant of a circle with center at B and $r = 2$ ft.

53. AB is a side of the square. Call the point of intersection of its diagonals, point O. $\overset{\frown}{AD}$ has B as its center and radius AB; $\overset{\frown}{DC}$ has O as its center and radius OD; $\overset{\frown}{BC}$ has A as its center and radius AB; the remaining arc is described on AB as a diameter.

54. Within the lines AB and CD, draw two lines a and b parallel respectively to AB and CD. The bisector of the angle formed by a and b is also the bisector of the angle found at the intersection of AB and CD. This line is the locus of the centers of the circles tangent to AB and CD.

55. For any circle, find such a line as OH. By using OH as a chord, divide the circle into ten equal arcs, calling the points of division 1, 2, 3, . . . , 10. By joining the even-numbered points of division a regular inscribed pentagon is formed; the diagonals of such a pentagon is a five-pointed star.

58. The triangle balances because the intersection of the medians is the center of gravity and the center of mass of the triangle.

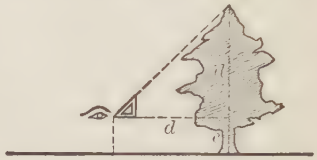
59. The successive diameters of the smaller ends of four pieces are 10″, 13″, 16″, and 19″; since the side of an inscribed square in circle of diameter D is $\frac{D}{2}\sqrt{2}$, the cross-sections of the timbers are

squares whose sides are respectively $5\sqrt{2}$, $\frac{13}{2}\sqrt{2}$, $8\sqrt{2}$, and $\frac{19}{2}\sqrt{2}$.

60. The bases of the trapezoid are $2R\pi$ and $2r\pi$; the altitude is $R - r$. The area equals $\frac{1}{2}(R - r)(2R\pi + 2r\pi) = \pi(R + r)(R - r)$.

61. The depth of the thread is the altitude of an equilateral triangle whose side is the pitch of the thread; the depth equals one-half the pitch times $\sqrt{3}$ or $d = \frac{1}{2}p \cdot \sqrt{3} = .866\,p$.

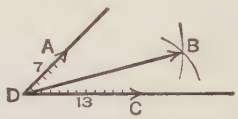
62. The triangle should be held so that one leg is vertical and so that a line sighted along the hypotenuse meets the top of the tree. The height of the eye from the ground plus the distance of the tree from the point of observation equals the height of the tree.



63. The pressure against the billboard $= 12 \times 50 \times .005 \times 30^2 = 2700$ lbs.

64. The areas of the trapezoids are $\frac{1}{2} \cdot 8(2 + 4)$ and $\frac{1}{2} \cdot 2(1 + 3)$; the areas of the triangles are $\frac{1}{2} \times 3 \times 2$; $\frac{1}{2} \times 4 \times 5$; $\frac{1}{2} \times 3 \times 7$; $\frac{1}{2} \times 1 \times 7$. The total area equals 55.

65.



66. The center of the triangular plate is the intersection of two of its medians; the center of the hexagonal plate is the intersection of two of its diagonals.

67. The method of Thucydides is correct for regular polygons only because the necessary dimensions for finding their areas may be found from their perimeters.

68. The diagonal of a rectangle 3×6 is $\sqrt{9 + 36} = 3\sqrt{5} = 6.708$. The table top can not be taken through the window.

69. The center of mass of the circular piece of cardboard is its center and since this point is below the point of support it is somewhere on AB. If some other point of support (A') be taken and in

similar manner another line be marked upon the circle, their point of intersection is the center of the circle.

70. If D = diameter of the earth in inches, its circumference = πD. The new circumference = $\pi (D + 1) = \pi D + \pi$. Therefore the circumference is increased 3.1416 inches.

71. There are 15 of the two-inch wafers and 60 of the one-inch wafers. Since the wafers are of the same thickness, we may compare their surface areas. The surface area of the top of the two-inch wafers $= 2^2 \times \frac{\pi}{4} \times 3 \times 5 = 15\pi$; that of the one-inch wafers $= 1^2 \times \frac{\pi}{4} \times 6 \times 10 = 15\pi$; therefore, the boxes contain equal amounts of chocolate.

72. A corner is a right angle, if two points 6 ft. and 8 ft. from the corner and lying in the sides are 10 ft. apart across the corner. The court is rectangular, if its diagonals are equal.

73. The distance between T and S is .83 in.

74. For the purpose of this problem, the amounts of water is to be determined by the areas of the cross-sections of the pipe openings. Since a 12-inch circle contains four times the area of a 6-inch circle, a 12-inch pipe would supply the required amount of water.

75. The area of the cross-section of a one-inch round bar is .7854 sq. in.; the load it will hold is .7854 × 30,000 lbs. = 23,562 lbs.

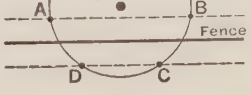
76. $\tan A = \frac{4}{7} = .5714$; $\tan 30° = .5774$ $\quad .5774$

$\tan 29° = .5543$ $\quad .5714$

$.0231$ $\quad .0060$

A difference of .0231 equals a difference of 1°; or we may say that we have a degree divided into 231 equal parts and we wish to find the value of 60 of them, or, $\frac{60}{231} \times 1° = 15' 35''$. The angle formed by the mechanic is 15′ 35″ too small, but for the majority of jobs would prove satisfactory.

77. A, B, C, and D represent the greatest number of possible solutions.



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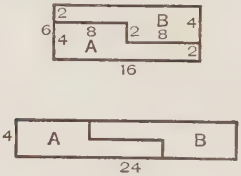
Recreations

1. Let $ABCD$ represent the original window. The hermit closed over one-half of it as shown in the figure, leaving the square opening $PRST$, which in one sense is one yard high and one yard wide since $PS = TR =$ one yard.

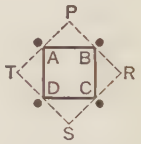
2. There are four corner posts with ten posts on each side between them or $4 + 4 \times 10 = 44$ posts.

3. Place two coins on a pile at each corner with one coin at any point of each side of the square.

4.



5.



6. The size or capacity of an empty glass is the same as the size or capacity of the same glass full of lemonade.

7. The first booth requires four strips; each succeeding booth requires three strips. If there are n booths, the number of strips is $4 + 3\,(n - 1) = 4 + 3n - 3 = 3n + 1$. Twelve booths require $3 \times 12 + 1 = 37$ strips.

8. From the formula of Ex. 7 the number of strips $= 3n + 1$; $3n + 1 = 49$; $n = 16$ booths.

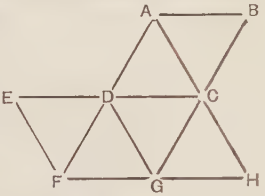
9.

10. The proof begs the question by assuming $\angle C > \angle B$ so that a part of $\angle C$ or $\angle 1$ equals $\angle B$.

11. The apparent proof is obtained by drawing an incorrect figure. The bisector of $\angle A$ and the perpendicular bisector of BC do not meet at O as shown. Let AO meet BC at R. Then $AC : AB = CR : RB$ (§ 286). If $AC > AB$ (as shown in the figure), $CR >$

RB and the bisector of $\angle A$ does not intersect *OP* within $\triangle ABC$ thus annulling the given proof. For a more complete discussion, see "Mathematical Recreations and Essays" by W. W. R. Ball, pp. 44 and 45.

12. The sum 34 is obtained in 4 rows, 4 columns, 2 diagonals, 1 large square, 9 small squares, 4 intermediate squares (such as 15, 3, 2, 14), 8 rectangles, 8 trapezoids, and 8 parallelograms or in a total of 48 ways.



13. In the first case remove *DC*, *CG*, and *DG;* in the second case remove *DF*, *DG*, *GH*, and *CH*.

14. Let the columns of a chessboard be designated by *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H* and the rows by 1, 2, 3, 4, 5, 6, 7, and 8. One arrangement of the eight pieces is A_2, B_5, C_7, D_1, E_3, F_8, G_6 and H_4. "Mathematical Recreations and Essays" by Ball gives eleven other solutions (p. 118).

15. Alice had a garden plot in the form of a right triangle whose altitude and base were respectively 3 rods and 4 rods. From this plot her father took away a section of two square rods and yet she found that the fence for the remaining plot was the same length as the original fence. Explain.

16. A solution is impossible because the triangle is impossible, since $39 + 45$ is not greater than 87 (§ 374).

17. The fallacy that 64 squares = 65 squares is due to the fact that in the latter case the sections do not make an exact fit. The variation is not easily detected, but the extremely narrow, long slit between the three sections is sufficient to equal an additional square so that 64 squares = 65 squares minus a slit equal to one square.

18. The fallacy is due to the fact that the triangles are not congruent under the relation *s.s.a.* = *s.s.a.* (See §125.)

19. The superintendent was right. If the rollers were on a fixed axis, the beams would move 4π in. with each revolution of the rollers, but since the rollers are not stationary, they also move 4π in. with each revolution; so the beam moves $4\pi + 4\pi = 8\pi = 25\frac{1}{7}$ in.

20. Draw a circle with six chords so that each chord intersects all the chords previously drawn. The following interesting series is thus developed:

No. of cuts	No. of pieces
0	$1 = 0 + 1 = 1$
1	$2 = 1 + 1 = 1 + 1$
2	$4 = 2 + 2 = 2 + 1 + 1$
3	$7 = 3 + 4 = 3 + 2 + 1 + 1$
4	$11 = 4 + 7 = 4 + 3 + 2 + 1 + 1$
5	$16 = 5 + 11 = 5 + 4 + 3 + 2 + 1 + 1$
6	$22 = 6 + 16 = 6 + 5 + 4 + 3 + 2 + 1 + 1$
7	$29 = 7 + 22 = 7 + 6 + 5 + 4 + 3 + 2 + 1 + 1$
n	$\frac{n(n+1)}{2} + 1 = n + \frac{n(n-1)}{2} + 1$

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